CONTENTS

▸ VECTOR & THREE DIMENSIONAL GEOMETRY

JEE (ADVANCED) SYLLABUS

Vector : Addition of vectors, scalar multiplication, dot and cross products, scalar triple products and their geometrical interpretations.
Three dimensions : Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

JEE (MAIN) SYLLABUS

Vector : Vectors and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.
3-D : Coordinates of a point in space, distance between two points, section formula, direction ratios and direction cosines angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equations of a line and a plane in different forms, intersection of a line and a plane, coplanar lines.

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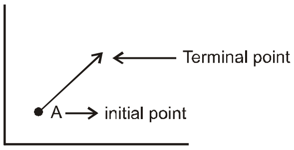
Vectors & Three Dimensional Geometry

What if angry vectors veer Round your sleeping head, and from. There's never need to fear Violence of the poor world's abstract storm. .... Warren, Robert PennNature is an infinite sphere of which the centre is everywhere and the circumference nowhere .... Pascal, Blaise

Vectors and their representation :



Vector quantities are specified by definite magnitude and definite direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** and B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

Zero vector :

A vector of zero magnitude i.e. which has the same initial and terminal point, is called a **zero vector.** It is denoted by **O. The direction of zero vector is indeterminate.**

Unit vector :

A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\vec{a}$, symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

Equal vectors :

Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.

Collinear vectors :

Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction($\rightrightarrows$) they are named as **like vectors** but if they have opposite direction ($\leftrightarrows$) then they are named as **unlike vectors**.

Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

$$\vec{a} = \lambda\vec{b} \Leftrightarrow \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\ (= \lambda)$$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Note : If $\vec{a}, \vec{b}$ are non zero, non–collinear vectors, such that $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'$, $y = y'$,
(where x, x', y, y' are scalars)

Example # 1 : Find unit vector of $\hat{i}-2\hat{j}+3\hat{k}$

Solution : $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ then $|\vec{a}| = \sqrt{a_x^2+a_y^2+a_z^2}$

$\therefore$ $|\vec{a}| = \sqrt{14}$ $\Rightarrow$ $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

Example # 2 : $\vec{a} = (x + 1)\hat{i} - (2x + y)\hat{j} + 3\hat{k}$ and $\vec{b} = (2x - 1)\hat{i} + (2 + 3y)\hat{j} + \hat{k}$

find $\vec{x}$ and $\vec{y}$ for which $\vec{a}$ and $\vec{b}$ are parallel.

Solution : $\vec{a}$ and $\vec{b}$ are parallel $\Rightarrow$ $\frac{x+1}{2x-1} = \frac{-(2x+y)}{2+3y} = \frac{3}{1}$ $\Rightarrow$ x = 4/5 , y = –19/25

Coplanar vectors :

A given number vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"two vectors are always coplanar"**.

Multiplication of a vector by a scalar :

If $\vec{a}$ is a vector and m is a scalar, then m is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called **scalar multiplication**. If $\vec{a}$ and $\vec{a}$ are vectors and m, n are scalars, then :

(i) $m\,(\vec{a}) = (\vec{a})\,m = m\vec{a}$ (ii) $m\,(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\,\vec{a} = m\vec{a} + n\vec{a}$ (iv) $m\,(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

Self Practice Problems :

(1) Given a regular hexagon ABCDEF with centre O, show that
(i) $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OE}$ (ii) $\overrightarrow{EA} = 2\,\overrightarrow{OB} + \overrightarrow{OF}$ (iii) $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} = 4\,\overrightarrow{AB}$

(2) Let ABCDEF be a regular hexagon. If $\overrightarrow{AD} = x\,\overrightarrow{BC}$ and $\overrightarrow{CF} = y\,\overrightarrow{AB}$ then find xy.

(3) The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

Answers : (2) – 4

Addition of vectors :

(i) If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

(ii) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ **(commutative)** (iii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ **(associative)**

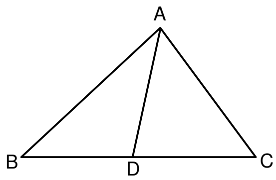
(iv) $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$ (v) $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

(vi) $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ (vii) $|\vec{a}-\vec{b}| \ge ||\vec{a}|-|\vec{b}||$

Example # 3 : The two sides of ΔABC are given by $\overrightarrow{AB} = 2\hat{i} + 4\hat{j} + 4\hat{k}$, $\overrightarrow{AC} = 2\hat{i} + 2\hat{j} + \hat{k}$. Then find the length of median through A.

Solution : Let D be mid point of BC

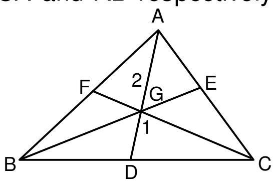
In ΔABC, $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ $\Rightarrow$ $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AD}$



$\Rightarrow$ $\frac{\overrightarrow{AB}+(\overrightarrow{AB}+\overrightarrow{BC})}{2} = \overrightarrow{AD}$

$\Rightarrow$ $\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2} = \overrightarrow{AD}$ $\Rightarrow |\overrightarrow{AD}| = \left|\frac{4\hat{i}+6\hat{j}+5\hat{k}}{2}\right| = \frac{\sqrt{77}}{2}$

Example # 4 : In a triangle ABC, D, E, F are the mid-points of the sides BC, CA and AB respectively then prove that, $\overrightarrow{AD} = -(\overrightarrow{BE} + \overrightarrow{CF})$.



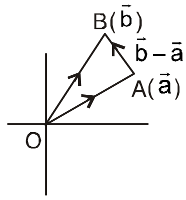
Solution : $\overrightarrow{AD} = 3\overrightarrow{GD} = 3.\frac{1}{2}(\overrightarrow{GB} + \overrightarrow{GC})$ where D is mid-point of BC

$= \frac{3}{2}\left[\frac{2}{3}\overrightarrow{EB} + \frac{2}{3}\overrightarrow{FC}\right] = -(\overrightarrow{BE} + \overrightarrow{CF})$

Position vector of a point:

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then

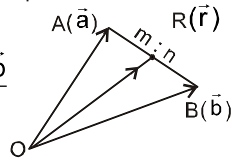
$\overrightarrow{AB} = \vec{b} - \vec{a}$ = position vector **(p.v.)** of B – position vector **(p.v.)** of A.



DISTANCE FORMULA

Distance between the two points A $(\vec{a})$ and B$(\vec{b})$ is AB = $|\vec{a} - \vec{b}|$

SECTION FORMULA



If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A (x_1, y_1, z_1) and B(x_2, y_2, z_2),

then the p.v. of a point R which divides AB in the ratio m: n is given by $\vec{r} = \frac{n\vec{a} + m\vec{b}}{m+n}$

Here R = $\left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}, \frac{nz_1 + mz_2}{n+m}\right)$

Note : Position vector of mid point M of AB is $\frac{\vec{a} + \vec{b}}{2}$. Here M = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Example # 5 : Let O be the centre of a regular pentagon ABCDE and $\overrightarrow{OA} = \vec{a}$.
Then $\overrightarrow{AB} + 2\overrightarrow{BC} + 3\overrightarrow{CD} + 4\overrightarrow{DE} + 5\overrightarrow{EA} =$

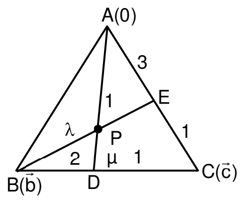
Solution : $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}, \overrightarrow{OD} = \vec{d}, \overrightarrow{OE} = \vec{e}$

$\overrightarrow{AB} + 2\overrightarrow{BC} + 3\overrightarrow{CD} + 4\overrightarrow{DE} + 5\overrightarrow{EA} = (\vec{b} - \vec{a}) + 2(\vec{c} - \vec{b}) + 3(\vec{d} - \vec{c}) + 4(\vec{e} - \vec{d}) + 5(\vec{a} - \vec{e})$

$= 5\vec{a} - (\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}) = 5\vec{a}$, (since $\vec{a} + \vec{b} + \vec{c} + \vec{d} + +\vec{e} = 0$)

Example # 6 : In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find $\frac{BP}{PE}$ using vector method.

Solution : Let the position vectors of points B and C be respectively $\vec{b}$ and $\vec{c}$ referred to A as origin of reference.



Let $\frac{BP}{PE} = \lambda$ and $\frac{PD}{AP} = \mu$

$\overrightarrow{AD} = \frac{2\vec{c} + \vec{b}}{3}$, $\overrightarrow{AE} = \frac{3}{4}\vec{c} \Rightarrow \overrightarrow{AP} = \frac{\frac{3\lambda\vec{c}}{4} + \vec{b}}{\lambda + 1} = \frac{\frac{2\vec{c} + \vec{b}}{3}}{\mu + 1}$

comparing the coefficient of $\vec{b}$ & $\vec{c}$

$\frac{1}{\lambda + 1} = \frac{1}{3(\mu + 1)}$ and $\frac{3\lambda}{4(\lambda + 1)} = \frac{2}{3(\mu + 1)}$

solving above equations we get $\lambda = 8/3$

Self Practice Problems

(4) Express vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

(5) If $\vec{a}, \vec{b}$ are position vectors of the points(1,–1),(–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

(6) The vertices P, Q and S of a ΔPQS have position vectors $\vec{p}, \vec{q}$ and $\vec{s}$ respectively.

(i) Find the position vector of $\vec{t}$ of point T in terms of $\vec{p}, \vec{q}$ and $\vec{s}$, such that ST : TM = 2 : 1 and M is mid-point of PQ.

(ii) If the parallelogram PQRS is now completed. Express , the position vector of the point R in terms of $\vec{p}, \vec{q}$ and $\vec{s}$

(7) In a quadrilateral ABCD, $\overrightarrow{AB} = \vec{p}$, $\overrightarrow{BC} = \vec{q}$, $\overrightarrow{DA} = \vec{p} - \vec{q}$. If E is the mid point of BC and F is the point on DE such that DF = $\frac{4}{5}$ DE. Show that the points. A,F,C are collinear.

(8) Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio 1 : 3.

Answers : (4) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ (5) m = 2

(6) (i) $\vec{t} = \frac{1}{3}(\vec{p}+\vec{q}+\vec{s})$ (ii) $\vec{r} = (\vec{q}-\vec{p}+\vec{s})$

Distance formula

Distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

Distance of a point P from coordinate axes

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then PA = $\sqrt{y^2+z^2}$, PB = $\sqrt{z^2+x^2}$, PC = $\sqrt{x^2+y^2}$

Example # 7 : Find the locus of a point which is equidistance from A (0,2,3) and B (2, –2, 1).

Solution : let P (x, y, z) be any point which is equidistance from A (0,2,3) and B (2, – 2, 1)

PA = PB

$\Rightarrow \sqrt{(x-0)^2+(y-2)^2+(z-3)^2} = \sqrt{(x-2)^2+(y+2)^2+(z-1)^2} \Rightarrow x - 2y - z + 1 = 0$

Example # 8 : Find the locus of a point which moves such that the sum of its distances from points A(0, 0, – α) and B(0, 0, α) is constant.

Solution : Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$\therefore \sqrt{(x-0)^2+(y-0)^2+(z+\alpha)^2} + \sqrt{(x-0)^2+(y-0)^2+(z-\alpha)^2} = 2a$

$\Rightarrow \sqrt{x^2+y^2+(z+\alpha)^2} = 2a - \sqrt{x^2+y^2+(z-\alpha)^2}$

$\Rightarrow x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha - 4a\sqrt{x^2+y^2+(z-\alpha)^2}$

$\Rightarrow 4z\alpha - 4a^2 = -4a\sqrt{x^2+y^2+(z-\alpha)^2} \Rightarrow \frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$

or, $x^2 + y^2 + z^2\left(1-\frac{\alpha^2}{a^2}\right) = a^2 - \alpha^2 \Rightarrow \frac{x^2}{a^2-\alpha^2} + \frac{y^2}{a^2-\alpha^2} + \frac{z^2}{a^2} = 1$

This is the required locus.

Self practice problems :

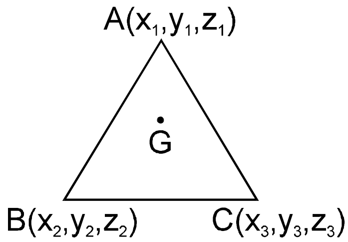
(9) One of the vertices of a cuboid is (0, 2, –1) and the edges from this vertex are along the positive x-axis, positive y-axis and positive z-axis respectively and are of lengths 2, 2, 3 respectively find out the vertices.

(10) Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

(11) Find the locus of point P if $AP^2–BP^2=20$, where A ≡ (2, –1, 3) and B ≡ (–1, – 2, 1).

Answers : (9) (2,2, –1), (2, 4, –1), (2, 4, 2), (2, 2, 2), (0, 2, 2), (0, 4, 2), (0, 4, –1).
(11) x + y + 2z = 6

Centroid of a triangle



$$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Incentre of triangle ABC

$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right)$$ Where AB = c, BC = a, CA = b

Example # 9 : Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Solution : Given A ≡ (2, 3, 4), B ≡ (–1, 2, –3), C ≡ (– 4, 1, –10).

Let C divide AB internally in the ratio k : 1, then

$C \equiv \left(\frac{-k+2}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+4}{k+1}\right)$ ∴ $\frac{-k+2}{k+1} = -4$ ⇒ 3k = – 6 ⇒ k = –2

For this value of k, $\frac{2k+3}{k+1} = 1$, and $\frac{-3k+4}{k+1} = -10$

Since k < 0, therefore C divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

Example # 10 : The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of ∠BAC meets BC in D. Find AD.

Solution : $AB = \sqrt{4^2+5^2+3^2} = 5\sqrt{2}$

$AC = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}$

Since AD is the internal bisector of BAC

∴ $\frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3}$ ∴ D divides BC internally in the ratio 5 : 3

∴ $D \equiv \left(\frac{5\times4+3\times1}{5+3}, \frac{5\times3+3(-1)}{5+3}, \frac{5\times2+3\times3}{5+3}\right)$ or, $D = \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$

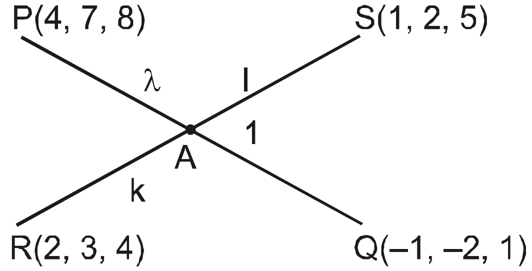
∴ $AD = \sqrt{\left(5-\frac{23}{8}\right)^2+\left(4-\frac{12}{8}\right)^2+\left(6-\frac{19}{8}\right)^2} = \frac{\sqrt{1530}}{8}$ unit

Example # 11 : If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1,2,5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Solution : Let the lines PQ and RS intersect at point A.

Let A divide PQ in the ratio λ : 1, (λ ≠ –1) then $A \equiv \left(\frac{-\lambda+4}{\lambda+1},\ \frac{-2\lambda+7}{\lambda+1},\ \frac{\lambda+8}{\lambda+1}\right)$. .... (1)

Let A divide RS in the ratio k : 1, then $A \equiv \left(\frac{k+2}{k+1},\ \frac{2k+3}{k+1},\ \frac{5k+4}{k+1}\right)$ ..... (2)



From (1) and (2), we have,

$\frac{-\lambda+4}{\lambda+1} = \frac{k+2}{k+1}$ ⇒ – λk – λ + 4k + 4 = λk + 2λ + k + 2 ⇒ 2λk + 3λ – 3k – 2 = 0 .....(3)

$\frac{-2\lambda+7}{\lambda+1} = \frac{2k+3}{k+1}$ ⇒ –2λk – 2λ + 7k + 7 = 2λk + 3λ + 2k + 3 ⇒ 4λk + 5λ – 5k – 4 = 0 .....(4)

$\frac{\lambda+8}{\lambda+1} = \frac{5k+4}{k+1}$ ..... (5)

Multiplying equation (3) by 2, and subtracting from equation (4), we get – λ + k = 0 or, λ = k

Putting λ = k in equation (3), we get $2\lambda^2 + 3\lambda - 3\lambda - 2 = 0$ ⇒ λ = 1 = k

Clearly λ = k = 1 satisfies eqn. (5), hence our assumption is correct.

∴ $A \equiv \left(\frac{-1+4}{2},\ \frac{-2+7}{2},\ \frac{1+8}{2}\right)$ or, $A \equiv \left(\frac{3}{2},\ \frac{5}{2},\ \frac{9}{2}\right)$.

Self practice problems :

(12) Find the ratio in which yz plane divides the line joining the points A (4, 3, 5) and B (7, 4, 5).

(13) Find the co-ordinates of the foot of perpendicular drawn from the point A(1, 2, 1) to the line joining the point B(1, 4, 6) and C(5, 4, 4).

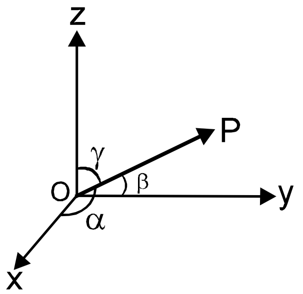
(14) Two vertices of a triangle are (4, –6, 3) and (2, –2, 1) and its centroid is $\left(\frac{8}{3},\ -1,\ 2\right)$. Find the third vertex.

(15) Show that $\left(\frac{3}{2},\frac{7}{2},\frac{1}{2}\right)$ is the circumcentre of the triangle whose vertices are A (2, 3, 2), B (0, 4, 1) and C (3, 3, 0) and hence find its orthocentre.

Answers : (12) 4 : 7 Externally (13) (3, 4, 5) (14) (2, 5, 2) (15) (2, 3, 2)

Direction cosines and direction ratios

(i) **Direction cosines :** Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by ℓ, m, n.



Thus ℓ = cos α, m = cos β, n = cos γ.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) **Direction ratios :** Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L, then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L, then $\ell\ \hat{i} + m\ \hat{j} + n\ \hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then (ℓ, m, n)

$$= \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) \text{or} \left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right)$$

(v) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

Example # 12 : If a line makes angle α, β, γ with the co-ordinate axes. Then find the value of $\sum \frac{\cos 3\alpha}{\cos\alpha}$.

Solution : $\sum \frac{\cos 3\alpha}{\cos\alpha} = \sum \frac{4\cos^3\alpha - 3\cos\alpha}{\cos\alpha} = \Sigma(4\cos^2\alpha - 3)$

$= 4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 - 3 - 3 = 4 - 9 = -5$ **Ans.** –5

Example # 13 : If the direction ratios of two lines are given by $mn - 4n\ell + 3\ell m = 0$ and $\ell + 2m + 3n = 0$ then find the direction ratios of the lines.

Solution : Eliminating ℓ we have $m = \pm\sqrt{2}\ n$ $\therefore$ $\frac{\ell}{-2\sqrt{2}-3} = \frac{m}{\sqrt{2}} = \frac{n}{1}$ & $\frac{\ell}{2\sqrt{2}-3} = \frac{m}{-\sqrt{2}} = \frac{n}{1}$

Ans. $((-2\sqrt{2}-3)\lambda, \sqrt{2}\lambda, \lambda), (2\sqrt{2}-3)\lambda,\ \sqrt{2}\lambda - \lambda$ where $\lambda \in R - \{0\}$

Self practice problems:

(16) Find the direction cosines of a line lying in the xy plane and making angle 30° with x-axis.

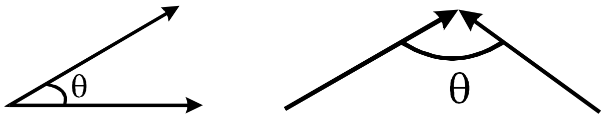
(17) A line makes an angle of 60° with each of x and y axes, find the angle which this line makes with z-axis.

(18) A plane intersects the co-ordinates axes at point A(2, 0, 0), B(0, 4, 0), C(0, 0, 6) ; O is origin. Find the direction ratio of the line joining the vertex B to the centroid of face ABC.

Answers : (16) $\ell = \frac{\sqrt{3}}{2}$, $m = \pm\frac{1}{2}$, $n = 0$ (17) 45° (18) $\frac{2}{3}, -\frac{8}{3}, 2$

Angle between two vectors :

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. Note that $0^\circ \le \theta \le 180^\circ$.



Scalar product (Dot Product) of two vectors :

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta,\ (0 \le \theta \le \pi)$$

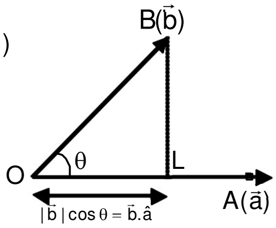
Note: (a) If θ is acute, then $\vec{a}\,.\,\vec{b} > 0$ and if θ is obtuse, then $\vec{a}\,.\,\vec{b} < 0$.

(b) $\vec{a}\,.\,\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0,\ \vec{b} \ne 0)$

(c) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}|\,|\vec{b}|$ (d) Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}|\,|\vec{b}|$

Geometrical interpretation of scalar product :

As shown in Figure, projection of vector $\overrightarrow{OB}$ (or $\vec{b}$) along vector $\overrightarrow{OA}$ (or $\vec{a}$) is $OL = |\vec{b}|\cos\theta = \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \vec{b}.\hat{a}$



Properties of Dot Product

(i) Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\,.\,\vec{b}}{|\vec{b}|}$

(ii) $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ **(commutative)**

(iii) $\vec{a}\,.\,(\vec{b}+\vec{c}) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$ **(distributive)**

(iv) $(m\vec{a})\,.\,\vec{b} = \vec{a}\,.(m\vec{b}) = m(\vec{a}\,.\,\vec{b})$, where m is a scalar.

(v) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vi) $\vec{a}\,.\,\vec{a} = |\vec{a}|^2 = \vec{a}^2$

(vii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\,.\,\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(viii) $|\vec{a} \pm \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$, where θ is the angle between the vectors

Example # 14 : Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Solution : (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}\ .\ \vec{b} = 0 \Rightarrow \left(3\hat{i} + 2\hat{j} + 9\hat{k}\right).\left(\hat{i} + p\hat{j} + 3\hat{k}\right) = 0$

$\Rightarrow \quad 3 + 2p + 27 = 0 \quad \Rightarrow \quad p = -15$

(ii) vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \quad \Rightarrow \quad 3 = \frac{2}{p} \quad \Rightarrow \quad p = \frac{2}{3}$$

Example # 15 : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that each is inclined at an angle $\pi/3$ with the other two and $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 3$, then find the scalar product of the vectors $2\vec{a} + 3\vec{b} - 5\vec{c}$ and $4\vec{a} - 6\vec{b} + 10\vec{c}$.

Solution : Dot products is $8^2\vec{a} - 18^2\vec{b} - 50^2\vec{c} + \vec{a}.\vec{b}\ (-12 + 12) + \vec{b}.\vec{c}\ (30 + 30) + \vec{c}\ .\ \vec{a}\ (20 - 20)$

$= 8 - 18\ (4) - 50(9) + 60\ \left(2.3\cos\frac{\pi}{3}\right) = 188 - 522 = -334$

Example # 16 : Find the values of x for which the angle between the vectors

$\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.

Solution : The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}||\vec{b}|}$

Now, θ is obtuse $\Rightarrow \cos\theta < 0 \Rightarrow \dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}||\vec{b}|} < 0 \Rightarrow \vec{a}\ .\ \vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow 14x^2 - 8x + x < 0 \Rightarrow 7x\ (2x - 1) < 0 \Rightarrow x(2x - 1) < 0 \Rightarrow 0 < x < \dfrac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Example # 17 : If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $= 2\vec{a} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$. (ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$.

Solution : (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}|^2}\right)\vec{a}$; Here $\vec{a}.\vec{b} = 2 - 1 + 3 = 4$ and $|\vec{a}|^2 = 3$

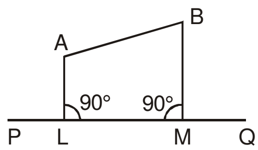
Hence $\left(\dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}|^2}\right)\vec{a} = \dfrac{4}{3}\vec{a} = \dfrac{4}{3}\ (\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$ is $\vec{b} - \left(\dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}|^2}\right)\vec{a}. = \dfrac{1}{3}\left(2\hat{i} - 7\hat{j} + 5\hat{k}\right)$

Example # 18 : Find the projection of the line joining A(1, 2, 3) and B(–1, 4, 2) on the line having direction ratios 2, 3, –6.

Solution : $\overrightarrow{AB} = -2\hat{i} + 2\hat{j} - \hat{k}$

Projection of $\overrightarrow{AB} = -2\hat{i} + 2\hat{j} - \hat{k}$ on $2\hat{i} + 3\hat{j} - 6\hat{k}$ is $\dfrac{-4 + 6 + 6}{\sqrt{4 + 9 + 36}} = \dfrac{8}{7}$



Self Practice Problems :

(19) If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is angle between them, prove that $\tan\frac{\theta}{2} = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$.

(20) Find the values of x for which the angle between the vectors $\vec{a} = 2x\,\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}-3\hat{j}+x\hat{k}$ is 90º

(21) if a, b, c are the pth, qth, rth terms of a HP then find the angle between the vectors $= (q-r)\,\hat{i} + (r-q)\,\hat{j} + (p-q)\,\hat{k}$ and $\vec{v} = \frac{1}{a}\,\hat{i} + \frac{1}{b}\,\hat{j} + \frac{1}{c}\,\hat{k}$.

(22) The points O, A, B, C, D are such that $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = 2\vec{a}+3\vec{b}, \overrightarrow{OD} = \vec{a}+2\vec{b}$
Given that the length of $\overrightarrow{OA}$ is three times the length of $\overrightarrow{OB}$. Show that $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular.

(23) ABCD is a tetrahedron and G is the centroid of the base BCD. Prove that
$AB^2 + AC^2 + AD^2 = GB^2 + GC^2 + GD^2 + 3GA^2$

(24) A (2, 3, –2), B (1, 5, 4,), C(0, –1, 2) D (4, 0, 3). Find the projection of line segment AB on CD line.

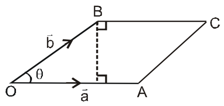
(25) The projections of a directed line segment on co-ordinate axes are 3, 4, –12. Find its length and direction cosines.

Answers : (20) x = 12/7 (21) $\pi/2$ (24) $\frac{2\sqrt{2}}{3}$ (25) 13, $\frac{3}{13},\frac{4}{13},\frac{-12}{13}$

Vector product (Cross Product) of two vectors:

(i) If $\vec{a}$, $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ **(not commutative)**

(iv) $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$, where m is a scalar.

(v) $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ **(distributive)**

(vi) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0,\ \vec{b}\neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(vii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(x) A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(xii) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of $\Delta ABC = \dfrac{1}{2}(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$. The points A, B and C are collinear if $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

(xiii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

Example # 19 : Given $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$, then find a unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$.

Solution : $\vec{\alpha} \times \vec{\beta}$ is $\perp$ to both $\vec{\alpha}$ and $\vec{\beta}$

$\vec{a} + \vec{b} = 3\hat{j}$, $\vec{b} + \vec{c} = -2\hat{i} + 4\hat{j}$

$\therefore$ $3\hat{j} \times (-2\hat{i} + 4\hat{j})$ is $\perp$ to both or $-6\hat{j} \times \hat{i} = 6\hat{k}$ $\therefore$ hence a unit vector is $\hat{k}$.

Example # 20 : If $\vec{\alpha} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{\beta} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{\gamma} = \hat{i} + \hat{j} + \hat{k}$, then find value $(\vec{\alpha} \times \vec{\beta}).(\vec{\alpha} \times \vec{\gamma})$.

Solution : $\vec{\alpha} \times \vec{\beta} = -10\hat{i} + 9\hat{j} + 7\hat{k}$ and $\vec{\alpha} \times \vec{\gamma} = 4\hat{i} - 3\hat{j} - \hat{k}$ their dot product = –40 – 27 – 7 = –74

Example # 21 : Let $\overrightarrow{OA} = \vec{a} + 3\vec{b}$, $\overrightarrow{OB} = 5\vec{a} + 4\vec{b}$ and $\overrightarrow{OC} = 2\vec{a} - \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Find $\dfrac{p}{q}$.

Solution : We have, p = Area of the quadrilateral OABC

$\Rightarrow$ $p = \dfrac{1}{2}|\overrightarrow{OB}\times\overrightarrow{AC}| = \dfrac{1}{2}|\overrightarrow{OB}\times(\overrightarrow{OC}-\overrightarrow{OA})|$ $\Rightarrow$ $p = \dfrac{1}{2}|(5\vec{a}+4\vec{b}) \times (4\vec{b}-\vec{a})|$

$\Rightarrow$ $p = \dfrac{1}{2}|20(\vec{a}\times\vec{b}) - 4(\vec{b}\times\vec{a})| = 12\ |\vec{a}\times\vec{b}|$ .....(i)

and q = Area of the parallelogram with OA and OC as adjacent sides

$\Rightarrow$ $q = |\overrightarrow{OA}\times\overrightarrow{OC}| = |(\vec{a}\times3\vec{b})\times(2\vec{a}-\vec{b})| = 7|\vec{a}\times\vec{b}|$ .....(ii)

From (i) and (ii), we get $\dfrac{p}{q} = \dfrac{12}{7}$

Self Practice Problems :

(26) If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30º. Find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

(27) ABC is a triangle and EF is any straight line parallel to BC meeting AC, AB in E, F respectively. If BR and CQ be drawn parallel to AC, AB respectively to meet EF in R and Q respectively, prove that Δ ARB = ΔACQ.

Answers : (26) 3/4 sq. units

A LINE

Equation of a line

(i) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar.

(ii) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(iii) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$. This form is called symmetric form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(iv) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(v) Reduction of cartesian form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow = \vec{r}\ (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda\,(a\hat{i} + b\hat{j} + c\hat{k}).$$

(vi) The equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are : $\vec{r} = \vec{a} + t\left(\hat{b} + \hat{c}\right)$ and $\vec{r} = \vec{a} + p\left(\hat{c} - \hat{b}\right)$.

Example # 22 : Find the equation of the line through the points (4, –5, 8) and (–1, 2, 7) in vector form as well as in cartesian form.

Solution : Let $A \equiv (4, -5, 8)$, $B \equiv (-1, 2, 7)$

Now $\vec{a} = \overrightarrow{OA} = 4\hat{i} - 5\hat{j} + 8\hat{k}$ and $\vec{b} = \overline{OB} = -\hat{i} + 2\hat{j} + 7\hat{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = 4\hat{i} - 5\hat{j} + 8\hat{k} + t(-5\hat{i} + 7\hat{j} - \hat{k})$ ..... (1)

Equation of AB in cartesian form is $\frac{x-4}{5} = \frac{y+5}{-7} = \frac{z-8}{1}$

Example # 23 : Find the equation of the line passing through point (1, 0, 2) having direction ratio 3, –1, 5. Prove that this line passes through (4, – 1, 7).

Solution. equation of line is $\frac{x-1}{3} = \frac{y-0}{-1} = \frac{z-2}{5}$,

Now $\frac{4-1}{3} = \frac{-1-0}{-1} = \frac{7-2}{5} = 1$, so line passes through point (4, –1,7)

Example # 24 : Find the equation of the line drawn through point (–1, 7, 0) to meet at right angles the line $\frac{x-2}{2}=\frac{y+3}{-1}=\frac{z-1}{2}$

Solution : Given line is $\frac{x-2}{2}=\frac{y+3}{-1}=\frac{z-1}{2}$ ..... (1)

Let P ≡ (–1, 7, 0)

Co-ordinates of any point on line (1) may be taken as Q ≡ (2r + 2, – r – 3, 2r + 1)

Direction ratios of PQ are 2r + 3, – r – 10, 2r + 1

Direction ratios of line AB are 2, – 1, 2

Since PQ $\perp$ AB

$\therefore$ 2(2r + 3) + (– r – 10) (–1) + 2 (2r + 1) = 0 $\Rightarrow$ r = – 2

Therefore, direction ratios of PQ are – 1 , – 8, – 3

Equation of line PQ is $\frac{x+1}{1}=\frac{y-7}{8}=\frac{z}{3}$

Example # 25 : A line passes through the point $3\hat{i}$ and is parallel to the vector $-\hat{i}+\hat{j}+\hat{k}$ and another line passes through the point $\hat{i}+\hat{j}$ and is parallel to the vector $\hat{i}+\hat{k}$, then find the point of intersection of lines.

Solution : A point on the first line is $3\hat{i}+s(-\hat{i}+\hat{j}+\hat{k})$ ....(i)

A point on the second line is $\hat{i}+\hat{j}+t(\hat{i}+\hat{k})$ ....(ii)

At the point of intersection (i) and (ii) are same.

$\therefore$ 3 – s = 1 + t, s = 1, s = t $\therefore$ s = t = 1

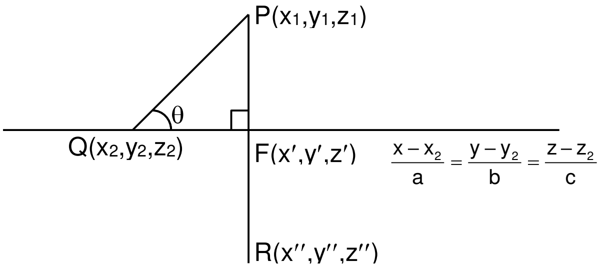
hence the point is $3\hat{i}+(-\hat{i}+\hat{j}+\hat{k})$ = $2\hat{i}+\hat{j}+\hat{k}$ Ans. (2,1,1)

Self practice problems:

(28) Find the equation of the line parallel to line $\frac{x-3}{4}=\frac{y+1}{-1}=\frac{z-7}{5}$ and passing through the point (2, 3, –2).

Answers : (28) $\frac{x-2}{4}=\frac{y-3}{1}=\frac{z+2}{5}$

Foot, Reflection, length of perpendicular from a point to a line :



Let L ≡ $\frac{x-x_2}{a}=\frac{y-y_2}{b}=\frac{z-z_2}{c}$ is a given line and P(x_1, y_1, z_1) is given point as shown in figure.

Let F(x′, y′, z′) = (ar+x_2, br+ y_2, cr+ z_2) ….(1) be the foot of the point P (x_1, y_1, z_1) with respect to the line L. Apply $\overrightarrow{PF}.(a\hat{i}+b\hat{j}+c\hat{k})=0$ we get 'r'. Now put this value of 'r' in (1) we get F

Now for calculating the reflection R(x″, y″, z″) of the point P (x_1, y_1, z_1) with respect to the line L, apply midpoint formula (midpoint of P & R is F)

$$PF = PQ \sin\theta = \left|\frac{\overrightarrow{PF}.(a\hat{i}+b\hat{j}+c\hat{k})}{\sqrt{a^2+b^2+c^2}}\right|$$

Example # 26 : Find the length of the perpendicular from P (2, – 3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$.

Solution : Co-ordinates of any point on given line may be taken as Q ≡ (2r – 1, 3r + 3, – r – 2)
Direction ratios of PQ are 2r – 3, 3r + 6, – r – 3
Direction ratios of AB are 2, 3, – 1
Since PQ ⊥ AB

$\therefore$ 2 (2r – 3) + 3 (3r + 6) – 1 (– r – 3) = 0 ⇒ 14r + 15 = 0 ⇒ $r = \frac{-15}{14}$

$\therefore$ $Q \equiv \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right)$ $\therefore$ $PQ = \sqrt{\frac{531}{14}}$ units.

Self practice problems :

(29) Find the length and foot of perpendicular drawn from point (2,3,4) to the line $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$. Also find the image of the point in the line.

Answers : (29) $3\sqrt{5}$, N ≡ (2, 6, –2), I ≡ (2, 9,–8)

Angle between two line :

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we can consider two vectors parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

(i) The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example # 27 : What is the angle between the lines whose direction cosines are

$-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$

Solution : Let θ be the required angle, then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{3}}{2}\right) . \left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2}$ ⇒ θ = 120°,

Example # 28 : P is a point on line $\vec{r} = 5\hat{i} + 7\hat{j} - 2\hat{k} + s(3\hat{i} - \hat{j} + \hat{k})$ and Q is a point on the line $\vec{r} = -3\hat{i} + 3\hat{j} + 6\hat{k} + t\left(-3\hat{i} + 2\hat{j} + 4\hat{k}\right)$. If $\overrightarrow{PQ}$ is parallel to the vector, $2\hat{i} + 7\hat{j} - 5\hat{k}$, find P and Q

Solution : $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ is parallel to $2\hat{i} + 7\hat{j} - 5\hat{k}$

$\therefore$ $-8\hat{i} - 4\hat{j} + 8\hat{k} + t(-3\hat{i} + 2\hat{j} + 4\hat{k}) - s\left(3\hat{i} - \hat{j} + \hat{k}\right) = 2\alpha = -8 - 3t - 3s$

7α = –4 + 2t + s ⇒ –5α = 8 + 4t – s

solving, α = t = s = –1

$\therefore$ P = (5, 7, –2) – (3, –1, 1) = (2, 8, –3) ⇒ Q = (–3, 3, 6) – (–3, 2, 4) = (0, 1, 2)

Self practice problems :

(30) Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$

(31) Let P (6, 3, 2), Q (5, 1, 4), R (3, 3, 5) are vertices of a Δ find $\angle Q$.

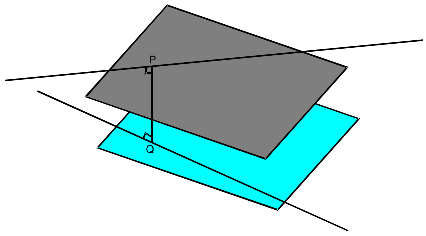
(32) Show that the direction cosines of a line which is perpendicular to the lines having directions cosines ℓ_1 m_1 n_1 and ℓ_2 m_2 n_2 respectively are proportional to

$m_1n_2 - m_2n_1$, $n_1\ell_2 - n_2\ell_1$, $\ell_1m_2 - \ell_2m_1$

Answers : (30) 60° (31) 90°

Skew Lines :

Lines in space which do not intersect and are also not parallel are called **skew line**.



If lines $\vec{r} = \vec{a} + \lambda_1\vec{p}$ & $\vec{r} = \vec{b} + \lambda_2\vec{q}$ are skew lines then $(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q}) \neq 0$

If lines are not skew lines then they are coplanar which means if $(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q}) = 0$, then lines are coplanar.

Shortest distance between two lines

(i) Shortest distance (d) between lines $\vec{r} = \vec{a} + \lambda_1\vec{p}$ & $\vec{r} = \vec{b} + \lambda_2\vec{q}$

is $d = \left| \dfrac{(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|} \right|$

(ii) Shortest distance (d) between two skew lines $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ and $\dfrac{x-\alpha'}{\ell'} = \dfrac{y-\beta'}{m'} = \dfrac{z-\gamma'}{n'}$

$$\text{is } d = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \div \sqrt{\sum (mn' - m'n)^2}$$

For Skew lines the direction of the shortest distance would be perpendicular to both the lines.

If d = 0, the lines are coplanar

(iii) Shortest distance between two parallel lines $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, is given by

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

Example # 29 : Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda\ (3\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu\ (-3\hat{i} + 2\hat{j} + 4\hat{k})$

Solution : Equation of given lines in cartesian form is $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$ (say L_1)

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$ (say L_2)

Let L on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$ and M on L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$

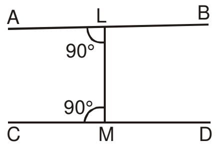
Direction ratios of LM are $3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$.

Since LM $\perp$ L_1

$\therefore$ $3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$ or, $11\lambda + 7\mu = 0$ ..... (1)

Again LM $\perp$ CD

$\therefore$ $-3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$ or, $-7\lambda - 29\mu = 0$ ..... (2)



Solving (1) and (2), we get $\lambda = 0, \mu = 0 \Rightarrow$ $L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t\ (6\hat{i} + 15\hat{j} - 3\hat{k})$

Note : Cartesian equation of LM is $\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$.

Self practice problems:

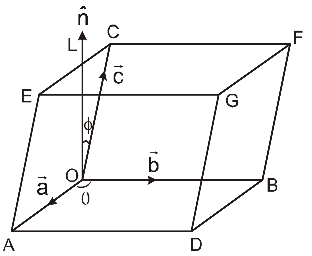
(33) Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Find also its equation.

Answers : (33) $\frac{1}{\sqrt{6}}$, $6x - y = 10 - 3y = 6z - 25$

Scalar triple product (Box Product) (S.T.P.) :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}|\ |\vec{b}|\ |\vec{c}|$, $\sin\theta \cdot \cos\phi$ where θ is the angle between $\vec{a}$, $\vec{b}$ **(i.e. $\vec{a} \wedge \vec{b} = \theta$)** and ϕ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ **($(\vec{a} \times \vec{b}) \wedge \vec{c} = \phi$)**. It is (i.e. $\vec{a} \times \vec{b} \cdot \vec{c}$) also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are



represented by $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $V = |[\vec{a}\ \vec{b}\ \vec{c}]|$

(iii) In a scalar triple product the position of dot and cross can be interchanged i.e.
$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{\ell}\ \vec{m}\ \vec{n}\right]$, where $\vec{\ell}$, $\vec{m}$ and $\vec{n}$ are non-coplanar vectors.

(vi) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vii) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(viii) $[\hat{i}\ \hat{j}\ \hat{k}] = 1$

(ix) $[K\vec{a}\ \vec{b}\ \vec{c}] = K\ [\vec{a}\ \vec{b}\ \vec{c}]$

(x) $[(\vec{a}+\vec{b})\vec{c}\vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(xi) $\left[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\right] = 0$ and $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

(xii) $\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Volume of Parallelopiped/ Tetrahedron and their properties :

(a) The volume of the parallelopiped whose three coterminous edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $V = [\vec{a}\ \vec{b}\ \vec{c}]$

(b) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right|$

(c) If the position vectors of the vertices of tetrahedron are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$.

Note : that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Example # 30 : The volume of the paralleopiped whose edges are represented by $-12\hat{i} + \lambda\hat{k}$, $3\hat{j} - k$, $2\hat{i} + \hat{j} - 15\hat{k}$ is 546, then find λ.

Solution : $V = \left|\begin{vmatrix} -12 & 0 & \lambda \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}\right| \quad \therefore \quad 546 = |12 \times 44 - 6\lambda| \quad \therefore \quad \lambda = -3, 179$

Example # 31 : Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Solution : Let four vertices be A, B, C, D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

$\therefore \quad \overrightarrow{DA} = (\vec{a} - \vec{d}) \Rightarrow \quad \overrightarrow{DB} = (\vec{b} - \vec{d}) \Rightarrow \overrightarrow{DC} = (\vec{c} - \vec{d})$

Hence volume V = $\frac{1}{6}$ $[\vec{a} - \vec{d} \quad \vec{b} - \vec{d} \quad \vec{c} - \vec{d}]$

$= \frac{1}{6}(\vec{a} - \vec{d}).[(\vec{b} - \vec{d}) \times (\vec{c} - \vec{d})] = \frac{1}{6}(\vec{a} - \vec{d}).[\vec{b} \times \vec{c} - \vec{b} \times \vec{b} + \vec{c} \times \vec{d}]$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{d}\ \vec{b}\ \vec{c}]\} = \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{b}\ \vec{c}\ \vec{d}]\}$

Example # 32 : Prove that vectors $\vec{r}_1 = (\sec^2 A, 1, 1)$; $\vec{r}_2 = (1, \sec^2 B, 1)$; $\vec{r}_3 = (1, 1, \sec^2 C)$ are always non-coplanar vectors if A, B, C $\in (0, \pi)$.

Solution : Condition of coplanarity gives D = 0 $\quad \Rightarrow \quad \begin{vmatrix} \sec^2 A & 1 & 1 \\ 1 & \sec^2 B & 1 \\ 1 & 1 & \sec^2 C \end{vmatrix} = 0$

$\Rightarrow \quad \sec^2 A\,[\sec^2 B \sec^2 C - 1] - 1(\sec^2 c - 1) + 1(1 - \sec^2 B) = 0$

$\Rightarrow \quad (1 + \tan^2 A)(\tan^2 B + \tan^2 C + \tan^2 B \tan^2 C) - \tan^2 C - \tan^2 B = 0$

$\Rightarrow \quad \tan^2 B \tan^2 C + \tan^2 A \tan^2 B + \tan^2 C \tan^2 A + \tan^2 A \tan^2 B \tan^2 C = 0$

divide by $\tan^2 A \tan^2 B \tan^2 C$

$\cot^2 A + \cot^2 B + \cot^2 C = -1$ it is a not possible

Example # 33 : If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular.

Solution : Let OABC be the tetrahedron, where O is the origin and co-ordinates of A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, x_3) respectively.

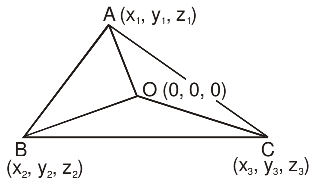
Let $\quad OA \perp BC$ and $OB \perp CA$.

We have to prove that $OC \perp BA$.

Now, direction ratios of OA are x_1, y_1, z_1 and of BC are $(x_3 - x_2), (y_3 - y_2), (z_3 - z_2)$.

$\because \quad OA \perp BC$ and $OB \perp CA$

$\Rightarrow \quad x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0 \quad$ and $\quad x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$



Adding above two equations we get $x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore \quad OC \perp BA$ ($\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$)

Self practice problems :

(34) Show that $\vec{a} \,.\, (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) = 0$

(35) One vertex of a parallelopiped is at the point A (1, –1, –2) in the rectangular cartesian co- ordinate. If three adjacent vertices are at B(–1, 0, 2), C(2, –2, 3) and D(4, 2, 1), then find the volume of the parallelopiped.

(36) Show that the vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$, $\vec{a}+\vec{b}$ are coplanar.

(37) Show that $\{(\vec{a}+\vec{b}+\vec{c}) \times (\vec{c}-\vec{b})\} \cdot \vec{a} = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(38) Find the value of m such that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+m\hat{j}+5\hat{k}$ are coplanar.

(39) Find the value of λ for which the four points with position vectors $-\hat{j}-\hat{k}$, $4\hat{i}+5\hat{j}+\lambda\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$, and $-4\hat{i}+4\hat{j}+4\hat{k}$ are coplanar.

Answer : (35) 72 (38) – 4 (39) $\lambda = 1$

Vector triple product :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product. This vector is perpendicular to $\vec{a}$ and lies in plane containing vectors $\vec{b}$ and $\vec{c}$

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$
- In general $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Example # 34 : $[\vec{a}\times(3\vec{b}+2\vec{c})\ \ \vec{b}\times(\vec{c}-2\vec{a})\ \ \ 2\vec{c}\times(\vec{a}-3\vec{b})] =$

Solution : Let $\vec{b}\times\vec{c} = \vec{p}$, $\vec{c}\times\vec{a} = \vec{q}$, $\vec{a}\times\vec{b} = \vec{r}$

$\therefore$ $[\vec{p}\ \vec{q}\ \vec{r}\,] = [\vec{a}\ \vec{b}\ \vec{c}\,]^2$ ....(i)

$\vec{a}\times(3\vec{b}+2\vec{c}) = 3\vec{r} - 2q$ etc.

$\therefore$ $E = [3\vec{r} - 2\vec{q}\ \ \vec{p} + 2\vec{r}\ ,\ 2\vec{q} + 6\vec{p}\,]$

$$= [0\vec{p} - 2\vec{q} + 3\vec{r}\ ,\ \vec{p} + 0\vec{q} + 2\vec{r}\ ,\ 6\vec{p} + 2\vec{q} + 0\vec{r}\,] = \begin{vmatrix} 0 & -2 & 3 \\ 1 & 0 & 2 \\ 6 & 2 & 0 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}\,]^2 = -18\,[\vec{a}\ \vec{b}\ \vec{c}\,]^2$$

Example # 35 : If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that $(\vec{a}\times\vec{b})\times\vec{c} = \left(\frac{\sqrt{3}\vec{b}+\vec{a}}{2}\right)$. Then find angles which makes $\vec{c}$ with $\vec{a}$ & $\vec{b}$ ($\vec{a}$ and $\vec{b}$ are non-collinear)

Solution : $(\vec{a}\times\vec{b})\times\vec{c} = \left(\frac{\sqrt{3}\vec{b}+\vec{a}}{2}\right)$ $\Rightarrow$ $(\vec{a}\cdot\vec{c}\,.),\vec{b} - (\vec{b}\cdot\vec{c})\vec{a} = \frac{\sqrt{3}\vec{b}+\vec{a}}{2}$

$\vec{a}\cdot\vec{c} = \frac{\sqrt{3}}{2}$, and $\vec{b}\cdot\vec{c} = -\frac{1}{2}$.

$\cos\theta = \frac{\sqrt{3}}{2}$ and $\cos\phi = -\frac{1}{2}$.

$\theta = \frac{\pi}{6}$ and $\phi = \frac{2\pi}{3}$

Example # 36 : Prove that $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}$ = $(\vec{b}\ .\ \vec{d})(\vec{a}\times\vec{c})$ – $(\vec{b}.\ \vec{c})$ $(\vec{a}\times\vec{d})$

Solution : We have, $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}$ = $\vec{a}\times\{(\vec{b}\ .\ \vec{d})\ \vec{c}-(\vec{b}\ .\ \vec{c})\ \vec{d}\}$

$= \vec{a}\times\{(\vec{b}\ .\ \vec{d})\ \vec{c}\}-\vec{a}\times\{(\vec{b}\ .\ \vec{c})\ \vec{d}\}$ [by dist. law]

$= (\vec{b}\ .\ \vec{d})\ (\vec{a}\times\vec{c})-(\vec{b}\ .\ \vec{c})\ (\vec{a}\times\vec{d})$.

Self Practice Problems :

(40) Prove that $\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\}=(\vec{a}\ .\ \vec{a})\ (\vec{b}\times\vec{a})$.

(41) Let $\vec{b}$ and $\vec{c}$ be noncollinear vectors. If $\vec{a}$ is a vector such that $\vec{a}\,.(\vec{b}+\vec{c})$ = 4 and $\vec{a}\times(\vec{b}\times\vec{c})$ = $(x^2 - 2x + 6)\ \vec{b}\ +\ \vec{c}$ siny, then find x and y.

(42) Find a unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$ is

Answer : (41) x = 1 & y = (4n + 1) π/2 , n $\in$ I (42) $\pm\left(\dfrac{\hat{j}-\hat{k}}{\sqrt{2}}\right)$

Linear combinations :

Given a finite set of vectors $\vec{a},\vec{b},\vec{c},......$, then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+........$ is called a linear combination of $\vec{a},\vec{b},\vec{c},......$ for any x, y, z..... $\in$ R. We have the following results :

(a) If $\vec{a},\vec{b}$ are non zero, non–collinear vectors, then $x\vec{a}+y\vec{b}=x'\vec{a}+y'\vec{b}\Rightarrow x=x'$, $y=y'$

(b) **Fundamental Theorem in plane :** Let $\vec{a},\vec{b}$ be non zero, non collinear vectors, then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$
i.e. there exist some unique x, y $\in$ R such that $x\vec{a}+y\vec{b}=\vec{r}$.

(c) If $\vec{a},\vec{b},\vec{c}$ are non–zero, non–coplanar vectors, then
$x\vec{a}+y\vec{b}+z\vec{c}=x'\vec{a}+y'\vec{b}+z'\vec{c}\Rightarrow x=x'\ ,\ y=y'\ ,\ z=z'$

(d) **Fundamental theorem in space:** Let $\vec{a},\vec{b},\vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i.e. there exist some unique x,y, z $\in$ R such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{r}$.

(e) If $\vec{x}_1,\vec{x}_2,......,\ \vec{x}_n$ are n non zero vectors and $k_1,k_2,.....,k_n$ are n scalars and if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+.......+k_n\vec{x}_n=\vec{0}\ \Rightarrow\ k_1=0,\ k_2=0\ ,\ .....,\ k_n=0$, then we say that vectors $\vec{x}_1,\ \vec{x}_2,\ ......,\vec{x}_n$ are **linearly independent vectors**.

(f) If $k_1\vec{x}_1+k_2\vec{x}_2+k_3\vec{x}_3......+k_r\vec{x}_r+......+k_n\vec{x}_n=\vec{0}$ and if there exists at least one $k_r\neq 0$, then $\vec{x}_1,\ \vec{x}_2,\ ......,\ \vec{x}_n$ are said to be **linearly dependent vectors.**
If $k_r\neq 0$ then $\vec{x}_r$ is expressed as a linear combination of vectors $\vec{x}_1,\ \vec{x}_2,........,\ \vec{x}_{r-1},\ \vec{x}_{r+1},\ ....,\ \vec{x}_n$

Note :

☞ In general, in 3 dimensional space every set of four vectors is a linearly dependent system.

☞ $\hat{i},\hat{j},\hat{k}$ are **Linearly Independent** set of vectors. For $K_1\hat{i}+K_2\hat{j}+K_3\hat{k}=\vec{0}\ \Rightarrow\ K_1=K_2=K_3=0$

☞ Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a}\times\vec{b}=\vec{0}\Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a}\times\vec{b}\neq\vec{0}$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

☞ If three vectors $\vec{a},\vec{b},\vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0$. Conversely if $[\vec{a}\ \ \vec{b}\ \ \vec{c}]\neq 0$ then the vectors are linearly independent.

Example # 37 : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors, solve the vector equation $\vec{r}.\vec{a} = \vec{r}.\vec{b} = \vec{r}.\vec{c} = 1$

Solution : since $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors therefore $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ & $\vec{c} \times \vec{a}$ are also non-coplanar vectors

Let $\vec{r} = x(\vec{a}\times\vec{b}) + y(\vec{b}\times\vec{c}) + z(\vec{c}\times\vec{a})$.

Then, $\vec{r}.\vec{a} = 1 \Rightarrow \quad 1 = y[(\vec{b}\times\vec{c})\ \vec{a}.]$

$y = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$, similarly $x = z = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} \Rightarrow \vec{r} = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}((\vec{a}\times\vec{b}) + (\vec{b}\times\vec{c}) + (\vec{c}\times\vec{a}))$

Example # 38 : Given that position vectors of points A, B, C are respectively

$\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$, $-7\vec{b} + 10\vec{c}$ then prove that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Solution : Let A, B, C be the given points and O be the point of reference then

$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\overrightarrow{OB} = 2\vec{a} + 3\vec{b} - 4\vec{c}$ and $\overrightarrow{OC} = -7\vec{b} + 10\vec{c}$

Now $\overrightarrow{AB}$ = p.v. of B – p.v. of A

$= \overrightarrow{OB} - \overrightarrow{OA} = (\vec{a} + 5\vec{b} - 7\vec{c})$ and $\overrightarrow{AC}$ = p.v. of C – p.v of A

$= \overrightarrow{OC} - \overrightarrow{OA} = -(\vec{a} + 5\vec{b} - 7\vec{c}) = -\overrightarrow{AB}$

$\therefore \quad \overrightarrow{AC} = \lambda\ \overrightarrow{AB}$

where $\lambda = -1$. Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Example # 39 : Prove that the vectors $5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$ are linearly dependent, where $\vec{a}, \vec{b}, \vec{c}$ being linearly independent vectors.

Solution : We know that if these vectors are linearly dependent , then we can express one of them as a linear combination of the other two.

Now let us assume that the given vector are coplanar, then we can write

$5\vec{a} + 6\vec{b} + 7\vec{c} = \ell(7\vec{a} - 8\vec{b} + 9\vec{c}) + m(3\vec{a} + 20\vec{b} + 5\vec{c})$ where ℓ, m are scalars

Comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ on both sides of the equation

$5 = 7\ell + 3m$, $\quad 6 = -8\ell + 20m$, $7 = 9\ell + 5m$

$\Rightarrow \ell = \frac{1}{2} = m$. Hence the given vectors are linearly dependent.

Self Practice Problems :

(43) Given that $\vec{x} + \frac{1}{\vec{p}^2}(\vec{p}\ .\ \vec{x})\ \vec{p} = \vec{q}$, show that $\vec{p}\ .\ \vec{x} = \frac{1}{2}\vec{p}\ .\ \vec{q}$ and find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

(44) If $\vec{x}\ .\ \vec{a} = 0$, $\vec{x}\ .\ \vec{b} = 0$ and $\vec{x}\ .\ \vec{c} = 0$ for some non-zero vector $\vec{x}$, then show that $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0$

(45) Prove that $\vec{r} = \frac{(\vec{r}\ .\ \vec{a})\ (\vec{b}\times\vec{c})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]} + \frac{(\vec{r}\ .\ \vec{b})\ (\vec{c}\times\vec{a})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]} + \frac{(\vec{r}\ .\ \vec{c})\ (\vec{a}\times\vec{b})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]}$

where $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors

(46) Does there exist scalars u, v, w such that $u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 = \hat{i}$ where $\vec{e}_1 = \hat{k}$, $\vec{e}_2 = \hat{j} + \hat{k}$, $\vec{e}_3 = -\hat{j} + 2\hat{k}$?

(47) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A} = (x + 4y)\,\vec{a} + (2x + y + 1)\,\vec{b}$ and $\vec{B} = (y - 2x + 2)\,\vec{a} + (2x - 3y - 1)\,\vec{b}$, find x and y such that $3\vec{A} = 2\vec{B}$.

(48) If vectors $\vec{a}, \vec{b}, \vec{c}$ be linearly independent, then show that

(i) $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - 4\vec{c}$, $-\vec{b} + 2\vec{c}$ are linearly dependent

(ii) $\vec{a} - 3\vec{b} + 2\vec{c}$, $-2\vec{a} - 4\vec{b} - \vec{c}$, $3a + 2\vec{b} - \vec{c}$ are linearly independent.

(49) Prove that a vector $\vec{r}$ in space can be expressed linearly in terms of three non-coplanar, non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ in the form

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}]\ \vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\ \vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\ \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Answers : (43) $\vec{x} = \vec{q} - \left(\frac{\vec{p}\,.\,\vec{q}}{2|\vec{p}|^2}\right)\vec{p}$ (46) No (47) $x = 2, y = -1$

Test of collinearity :

Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = 0 = \vec{0}$, where x + y + z = 0.

Test of coplanarity :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$, where x + y + z + w = 0.

Example # 40 : Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Solution : Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So let $\overrightarrow{PQ} = x\overrightarrow{PR} + y\overrightarrow{PS}$

$\Rightarrow$ $-\vec{a} - 5\vec{b} + 4\vec{c} = x(\vec{a} + \vec{b} - \vec{c}) + y(-\vec{a} - 9\vec{b} + 7\vec{c})$

$\Rightarrow$ $-\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\,\vec{a} + (x - 9y)\,\vec{b} + (-x + 7y)\,\vec{c}$

$\Rightarrow$ $x - y = -1$, $x - 9y = -5$, $-x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two equations of these three equations, we get $x = -\frac{1}{2}$, $y = \frac{1}{2}$.

These values also satisfy the third equation. Hence the given four points are coplanar.

Self Practice Problems :

(50) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that the terminal A, B, C, D of these vectors are coplanar. Find the point (P) at which AC and BD meet. Also find the ratio in which P divides AC and BD.

Answers : (50) $\vec{p} = \frac{3\vec{a} + \vec{c}}{4}$ divides AC in 1 : 3 and BD in 1 : 1 ratio

A PLANE

If line joining any two points on a surface lies completely on it then the surface is a plane.

OR

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

Equation of a plane :

(i) **Vector form :** The equation $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$ represents a plane containing the point with position vector is a vector normal to the plane.
The above equation can also be written as $\vec{r} \cdot \vec{n} = d$, where d = $\vec{r}_0 \cdot \vec{n}$

(ii) **Cartesian form :** The equation of a plane passing through the point (x_1, y_1, z_1) is given by
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(iii) **Normal form :** Vector equation of a plane normal to unit vector and at a distance d from the origin is $\vec{r} \cdot \vec{n} = d$. Normal form of the equation of a plane is $\ell x + my + nz = p$, where, ℓ,m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(iv) **General form :** $ax + by + cz + d = 0$ is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.

(v) **Plane through three points :** The equation of the plane through three non–collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$

(vi) **Intercept Form :** The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Note :

☞ Equation of yz–plane, xz–plane and xy–plane is x = 0, y = 0 and z = 0

☞ **Transformation of the equation of a plane to the normal form:** To reduce any equation $ax + by + cz - d = 0$ to the normal form, first write the constant term on the right hand side and make it positive, then divide each term by $\sqrt{a^2 + b^2 + c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{by}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{cz}{\pm\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\pm\sqrt{a^2 + b^2 + c^2}}$$

Where (+) sign is to be taken if d > 0 and (–) sign is to be taken if d < 0.

☞ A plane $ax + by + cz + d = 0$ divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$

☞ Coplanarity of four points

The points $A(x_1\, y_1\, z_1)$, $B(x_2\, y_2\, z_2)$ $C(x_3\, y_3\, z_3)$ and $D(x_4\, y_4\, z_4)$ are coplanar then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

Example # 41 : Find the equation of the plane upon which the length of normal from origin is 10 and direction ratios of this normal are 3, 2, 6.

Solution : If p be the length of perpendicular from origin to the plane and ℓ, m, n be the direction cosines of this normal, then its equation is

$\ell x + my + nz = 10$ ..... (1)

Direction ratios of normal to the plane are 3, 2, 6

$\therefore$ Direction cosines of normal to the required plane are $\ell = \frac{3}{7}$, $m = \frac{2}{7}$, $n = \frac{6}{7}$

Equation of required plane is $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$ or, $3x + 2y + 6z = 70$

Example # 42 : Find the plane through the points (2,–3,3), (–5, 2, 0), (1, – 7, 1)

Solution : $\begin{vmatrix} x-2 & y+3 & z-3 \\ -5-2 & 2+3 & 0-3 \\ 1-2 & -7+3 & 1-3 \end{vmatrix} = 0$ or $\begin{vmatrix} x-2 & y+3 & z-3 \\ -7 & 5 & -3 \\ -1 & -4 & -2 \end{vmatrix} = 0 \Rightarrow 2x + y - 3z + 8 = 0$

Example # 43 : If P be any point on the plane $\ell x + my + nz = p$ and Q be a point on the line OP such that $OP . OQ = p^2$, show that the locus of the point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Solution : Let $P \equiv (\alpha, \beta, \gamma)$, $Q \equiv (x_1, y_1, z_1)$

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have $\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k$ (say) ..... (1)

As P (α, β, γ) lies on the plane $\ell x + my + nz = p$,

$\ell\alpha + m\beta + n\gamma = p$ or $k(\ell x_1 + my_1 + nz_1) = p$ ..... (2)

Given $OP . OQ = p^2$ $\Rightarrow$ $\sqrt{\alpha^2 + \beta^2 + \gamma^2}\ \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$

or, $\sqrt{k^2(x_1^2 + y_1^2 + z_1^2)}\ \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$ or, $k\ (x_1^2 + y_1^2 + z_1^2) = p^2$ ....(3)

On dividing (2) by (3), we get $\frac{\ell x_1 + my_1 + nz_1}{x_1^2 + y_1^2 + z_1^2} = \frac{1}{p}$ or, $p\ (\ell x_1 + my_1 + nz_1) = x_1^2 + y_1^2 + z_1^2$

Hence the locus of point Q is $p\ (\ell x + my + nz) = x^2 + y^2 + z^2$.

Example # 44 : A moving plane passes through a fixed point (α,β,γ) and cuts the coordinate axes A, B, C . Find the locus of the centroid of the tetrahedron OABC.

Solution : Let the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, 0 (0,0,0) , A (a, 0,0), B (0, b, 0)

C (0,0,c) . Centroid of OABC is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$

The plane passes through (α,β,γ) $\therefore$ $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$ ......(i)

Centroid , $x = \frac{a}{4}$, $y = \frac{b}{4}$, $z = \frac{c}{4}$ or $a = 4x$, $b = 4y$, $c = 4z$

Now (1) gives the locus of G as $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 4$

Self practice problems :

(51) Check whether given points are coplanar if yes find the equation of plane containing them
$A \equiv (0, -1, -1)$, $B \equiv (4, 5, 1)$, $C \equiv (3, 9, 4)$, $D \equiv (-4, 4, 4)$

(52) Find the plane passing through point (3, 2, 1) and perpendicular to the line joining the points (2, 4, 3) and (3, –1, 5).

(53) Find the equation of plane passing through the point (2, 4, 6) and making equal intercepts on the coordinate axes.

(54) Find the equation of plane passing through (1, 2, –3) and (2, 3, 3) and perpendicular to the plane $2x + y - 3z + 4 = 0$.

(55) Find the equation of the plane parallel to $2\hat{i} + \hat{j} - \hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$ and passing through (2, 1, 3).

(56) Find the equation of the plane passing through the point (1, 1, – 1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

Answers :

(51)	yes, $5x - 7y + 11z + 4 = 0$	(52)	$x - 5y + 2z + 5 = 0$
(53)	$x + y + z = 12$	(54)	$9x - 15y + z + 24 = 0$
(55)	$x - y + z = 4$	(56)	$17x + 2y - 7z = 26$

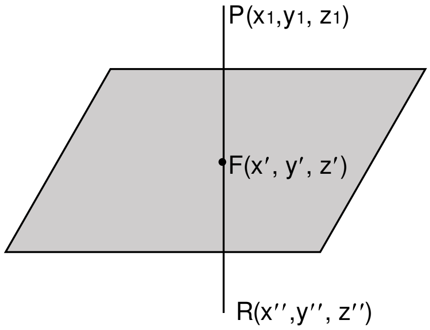
Position of point with respect to plane :

A plane divides the three dimensional space in two equal parts. Two points A $(x_1\ y_1\ z_1)$ and B $(x_2\ y_2\ z_2)$ are on the same side of the plane $ax + by + cz + d = 0$ if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

Example # 45 : Show that the points (1, 2, 3) and (2, – 1, 4) lie on opposite sides of the plane $x + 4y + z - 3 = 0$.

Solution : Since the numbers $1 + 4 \times 2 + 3 - 3 = 9$ and $2 - 4 + 4 - 3 = -1$ are of opposite sign, then points are on opposite sides of the plane.

A plane & a point



Let $P \equiv ax + by + cz + d = 0$ is a given plane and $P(x_1, y_1, z_1)$ is given point as shown in figure.

Let $F(x', y', z')$ be the foot of the point P (x_1, y_1, z_1) with respect to the plane P.

And $R(x'', y'', z'')$ be the reflection of point P (x_1, y_1, z_1) with respect to the plane P.

(i) Distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is given by $\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}.\vec{n} = d$ is $p = \frac{|\vec{a}\ .\ \vec{n} - d|}{|\vec{n}|}$.

(iii) The coordinates of the foot (F) of perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(iv) The coordinates of the Image (R) of point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = -\frac{2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

Example # 46 : Find the image of the point P (3, 5, 7) in the plane $2x + y + z = 0$.

Solution : Given plane is $2x + y + z = 0$ ..... (1)

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R then PQ $\perp$ plane (1)

Let $R \equiv (2r + 3, r + 5, r + 7)$

Since R lies on plane (1)

$\therefore$ $2(2r + 3) + r + 5 + r + 7 = 0$ or, $6r + 18 = 0$ $\therefore$ $r = -3$

$\therefore$ $R \equiv (-3, 2, 4)$

Let $Q \equiv (\alpha, \beta, \gamma)$

Since R is the middle point of PQ

$\therefore$ $-3 = \frac{\alpha + 3}{2} \Rightarrow \alpha = -9$ and $2 = \frac{\beta + 5}{2} \Rightarrow \beta = -1$ and $4 = \frac{\gamma + 7}{2} \Rightarrow \gamma = 1$

$\therefore$ $Q = (-9, -1, 1)$.

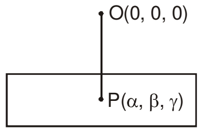
Example # 47 : A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$

Solution : Let the equation of the variable plane be $\ell x + my + nz + d = 0$ ..... (1)

Plane passes through the fixed point (a, b, c) $\therefore$ $\ell a + mb + nc + d = 0$ ..... (2)

Let P (α, β, γ) be the foot of perpendicular from origin to plane (1).

Direction ratios of OP are



$\alpha - 0, \beta - 0, \gamma - 0$ i.e. α, β, γ

From equation (1), it is clear that the direction ratios of normal to the plane i.e. OP are ℓ, m, n;

α, β, γ and ℓ, m, n are the direction ratios of the same line OP

$\therefore$ $\frac{\alpha}{\ell} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{1}{k}$ (say) $\therefore$ $\ell = k\alpha, m = k\beta, n = k\gamma$ ..... (3)

Putting the values of ℓ, m, n in equation (2), we get $ka\alpha + kb\beta + kc\gamma + d = 0$ ..... (4)

Since α, β, γ lies in plane (1) $\therefore$ $\ell\alpha + m\beta + n\gamma + d = 0$ ..... (5)

Putting the values of ℓ, m, n from (3) in (5), we get $k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ..... (6)

or $k\alpha^2 + k\beta^2 + k\gamma^2 - ka\alpha - kb\beta - kc\gamma = 0$ [putting the value of d from (4) in (6)]

or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of perpendicular P (α, β, γ) is $x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (7)

Self practice problems :

(57) Find the intercepts of the plane 3x + 4y – 7z = 84 on the axes. Also find the length of perpendicular from origin to this plane and direction cosines of this normal.

(58) Find : (i) perpendicular distance (ii) foot of perpendicular
(iii) image of (1, 1, 1) in the plane 3x + 4y – 12z + 13 = 0

Answers : (57) a = 28, b = 21, c = – 12, p = $\frac{84}{\sqrt{74}}$; $\frac{3}{\sqrt{74}}$, $\frac{4}{\sqrt{74}}$, $\frac{-7}{\sqrt{74}}$

(58) (i) $\frac{\sqrt{17}}{2}$ (ii) (–1, 1/2, 1) (iii) (–3, 0,1)

Angle between two planes :

(i) Consider two planes ax + by + cz + d = 0 and a′x + b′y + c′z + d′ = 0. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a′, b′, c′) respectively, hence θ, the angle between them, is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{a'^2 + b'^2 + c'^2}}$$

Planes are perpendicular if aa′ + bb′ + cc′ = 0 and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(ii) The angle θ between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\ |\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda \vec{n}_2$.

Distance between parallel planes :

Distance between two parallel planes ax + by + cz + d_1 = 0 and ax + by + cz + d_2 = 0 is $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

Example # 48 : Find the distance between the parallel planes 2x – y + 2z + 3 = 0 and 4x – 2y + 4z + 5 = 0

Solution : Given planes are 2x – y + 2z + 3 = 0 and 2x – y + 2z + 5/2 = 0

Required distance between planes = $\frac{|3 - 5/2|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} = \frac{1}{6}$

Angle bisectors

(i) The equations of the planes bisecting the angle between two given planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad ......(1)$$

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad ....(2)$$

(ii) If $a_1\alpha + b_1\beta + c_1\gamma + d_1$ and $a_2\alpha + b_2\beta + c_2\gamma + d_2$ are of same/opposite sign then (1)/(2) gives equation of angle bisector of region containing point (α, β, γ)

(iii) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then equation (1)/(2) gives obtuse/acute angle bisector and if $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then equation (1)/(2) gives acute/obtuse angle bisector.

Family of planes

(i) Any plane passing through the line of intersection of non–parallel planes or equation of the plane through the given line in non symmetric form.

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 + \lambda\ (a_2x + b_2y + c_2z + d_2) = 0$, where $\lambda \in R$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\ .\ \vec{n}_1 = d_1$ & $\vec{r}\ .\vec{n}_2 = d_2$ is. $\vec{r}\ (n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

Example # 49 : Find the equation of the plane through the line of intersection of the planes $x + 2y + 3z + 2 = 0$, $2x + 3y - z + 3 = 0$ and perpendicular to the plane $x + y + z = 0$

Solution : The plane is $x + 2y + 3z + 2 + \lambda\ (2x + 3y - z + 3) = 0$

or $(1 + 2\lambda)\ x + (2 + 3\lambda)\ y + (3 - \lambda)\ z + 2 + 3\lambda = 0$

It is perpendicular to $x + y + z = 0$

$\therefore\ 1 + 2\lambda + 2 + 3\lambda + 3 - \lambda = 0$ or $2\lambda + 3 = 0 \Rightarrow \lambda = -\dfrac{3}{2}$

Substituting we get $4x + 5y - 9z + 5 = 0$

Example # 50 : Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Solution : Given planes are $x + y + z - 6 = 0$ ..... (1)

and $2x + 3y + 4z + 5 = 0$ ..... (2)

Given point is P (1, 1, 1).

Equation of any plane through the line of intersection of planes (1) and (2) is

$x + y + z - 6 + k\ (2x + 3y + 4z + 5) = 0$ ..... (3)

If plane (3) passes through point P, then

$1 + 1 + 1 - 6 + k\ (2 + 3 + 4 + 5) = 0$ or, $k = \dfrac{3}{14}$

From (3) required plane is $20x + 23y + 26z - 69 = 0$

Example # 51 Let planes are $2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$. Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Solution : Given planes are $-2x - y - 2z + 9 = 0$ ..... (1)

and $3x - 4y + 12z + 13 = 0$ ..... (2)

Equations of bisecting planes are $\dfrac{-2x - y - 2z + 9}{\sqrt{(-2)^2 + (-1)^2 + (-2)^2}} = \pm \dfrac{3x - 4y + 12z + 13}{\sqrt{3^2 + (-4)^2 + (12)^2}}$

or, $13\ [-2x - y - 2z + 9] = \pm\ 3\ (3x - 4y + 12z + 13)$

or, $35x + y + 62z = 78$, ... (3) [Taking +ve sign]

and $17x + 25y - 10z = 156$ ..... (4) [Taking – ve sign]

Now $a_1a_2 + b_1b_2 + c_1c_2 = (-2)\ (3) + (-1)\ (-4) + (-2)\ (12)$

$= -6 + 4 - 24 = -26 < 0$

$\therefore$ Bisector of acute angle is given by $35x + y + 62z = 78$

$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Example # 52 : If the planes x – cy – bz = 0, cx – y + az = 0 and bx + ay – z = 0 pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Solution : Given planes are x – cy – bz = 0 ..... (1)

cx – y + az = 0 ..... (2)

bx + ay – z = 0 ..... (3)

Equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as $x - cy - bz + \lambda (cx - y + az) = 0$

or, $x (1 + \lambda c) - y (c + \lambda) + z (- b + a\lambda) = 0$ ..... (4)

If planes (3) and (4) are the same, then equations (3) and (4) will be identical.

$$\therefore \quad \underset{(i)}{\frac{1+c\lambda}{b}} = \underset{(ii)}{\frac{-(c+\lambda)}{a}} = \underset{(iii)}{\frac{-b+a\lambda}{-1}}$$

From (i) and (ii), $a + ac\lambda = - bc - b\lambda$

or, $\lambda = -\dfrac{(a+bc)}{(ac+b)}$ ..... (5)

From (ii) and (iii),

$c + \lambda = - ab + a^2\lambda$ or $\lambda = \dfrac{-(ab+c)}{1-a^2}$ ..... (6)

From (5) and (6), we have $\dfrac{-(a+bc)}{ac+b} = \dfrac{-(ab+c)}{(1-a^2)}$.

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

or, $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$ or, $a^2 + b^2 + c^2 + 2abc = 1$.

Self practice problems:

(59) Find the equation of plane passing through the line of intersection of the planes 2x – 7y + 4z = 3 and 3x – 5y + 4z = 11 and the point (–2, 1, 3).

(60) Find the equations of the planes bisecting the angles between the planes x + 2y + 2z – 3 = 0, 3x + 4y + 12z + 1 = 0 and sepecify the plane which bisects the acute angle between them.

(61) Show that the origin lies in the acute angle between the planes x + 2y + 2z – 9 = 0 and 4x – 3y + 12z + 13 = 0

(62) Prove that the planes 12x – 15y + 16z – 28 = 0, 6x + 6y – 7z – 8 = 0 and 2x + 35y – 39z + 12 = 0 have a common line of intersection.

Answers : (59) 15x – 47y + 28z = 7

(60) 2x + 7y – 5z = 21, 11x + 19y + 31z = 18; 2x + 7y – 5z = 21

Angle between a plane and a line:

(i) If θ is the angle between line $\dfrac{x-x_1}{\ell} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ and the plane ax + by + cz + d = 0, then

$$\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right].$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\,.\,\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\,.\,\vec{n}}{|\vec{b}|\,|\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\quad \frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} \quad \vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $\quad a\ell + bm + cn = 0 \quad \vec{b}\,.\,\vec{n} = 0$

Condition for a line to lie in a plane

(i) **Cartesian form:** Line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) **Vector form:** Line $\vec{r} = \vec{a} + \lambda\,\vec{b}$ would lie in the plane $\vec{r}\,.\,\vec{n} = d$ if $\vec{b}\,.\,\vec{n} = 0$ & $\vec{a}\,.\,\vec{n} = d$

Example # 53 : Find the distance of the point (1, 0, – 3) from the plane x – y – z = 9 measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

Solution : Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ..... (2)

Equation of a line passing through the point Q(1, 0, – 3) and parallel to line (2) is

$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r$. ..... (3)

Co-ordinates of any point on line (3) may be taken as

P (2r + 1, 3r, – 6r – 3)

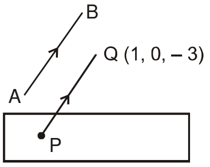
If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),

$\therefore$ (2r + 1) – (3r) – (– 6r – 3) = 9

r = 1

or, $P \equiv (3, 3, -9)$

Distance between points Q (1, 0, – 3) and P (3, 3, – 9)



$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(-3))^2} = \sqrt{4+9+36} = 7$.

Example # 54 : Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

Solution : Equation of any plane passing through (1, 2, 0) may be taken as

$a(x-1) + b(y-2) + c(z-0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point (– 3, 1, 2) on line (2) lies in plane (1)

$\therefore$ $a(-3-1) + b(1-2) + c(2-0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13}$ = k (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

$6(x-1) + 2(y-2) + 13(z-0) = 0.$

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Example # 55 : Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane $x + 2y + z = 9$.

Solution : Let the given line AB be $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ ..... (1)

Given plane is $x + 2y + z = 9$ ..... (2)

Let DC be the projection of AB on plane (2)

Clearly plane ABCD is perpendicular to plane (2).

Equation of any plane through AB may be taken as (this plane passes through the point (1, – 1, 3) on line AB)

$a(x-1) + b(y+1) + c(z-3) = 0$ ..... (3)

where $2a - b + 4c = 0$ ..... (4)

[$\because$ normal to plane (3) is perpendicular to line (1)]

Since plane (3) is perpendicular to plane (2),

$\therefore$ $a + 2b + c = 0$ ..... (5)

Solving equations (4) & (5), we get $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$.

Substituting these values of a, b and c in equation (3), we get

$9(x-1) - 2(y+1) - 5(z-3) = 0$

or, $9x - 2y - 5z + 4 = 0$ ...... (6)

Since projection DC of AB on plane (2) is the line of intersection of plane ABCD and plane (2), therefore equation of DC will be

$$\text{and} \left.\begin{aligned} 9x - 2y - 5z + 4 = 0 \quad .....(i) \\ x + 2y + z - 9 = 0 \quad .....(ii) \end{aligned}\right\} \quad ..... (7)$$

Let ℓ, m, n be the direction ratios of the line of intersection of planes (i) and (ii)

$\therefore$ $9\ell - 2m - 5n = 0$ ..... (8) and $\ell + 2m + n = 0$ ..... (9)

$\therefore$ $\frac{\ell}{-2+10} = \frac{m}{-5-9} = \frac{n}{18+2}$ $\Rightarrow$ $\frac{\ell}{4} = \frac{m}{-7} = \frac{n}{10}$

Let any point on line (7) is $(\alpha, \beta, 0)$ $\Rightarrow$ $9\alpha - 2\beta + 4 = 0$

$\alpha + 2\beta - 9 = 0$ $\Rightarrow$ $\alpha = \frac{1}{2}$, $\beta = \frac{17}{4}$

So equation of line is $\frac{x-\frac{1}{2}}{4} = \frac{y-\frac{17}{4}}{-7} = \frac{z-0}{10}$

Self practice problems :

(63) Find the values of a and b for which the line $\frac{x-2}{a}=\frac{y+3}{4}=\frac{z-6}{-2}$ is perpendicular to the plane $3x - 2y + bz + 10 = 0$.

(64) Find the equation of the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$.

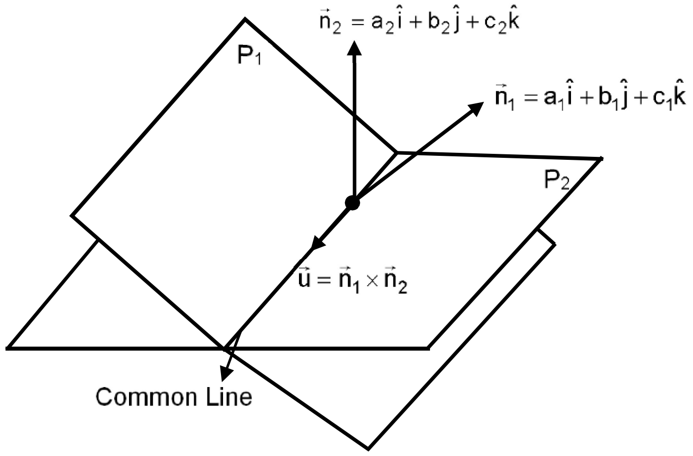
(65) Find the plane containing the line $\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{5}$ and parallel to the line $\frac{x+1}{1}=\frac{y-1}{-2}=\frac{-z+1}{1}$

(66) Show that the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ & $\frac{x-4}{5}=\frac{y-1}{2}=z$ are intersecting each other. Find their intersection point and the plane containing the line.

Answers : (63) $a = -6, b = 1$ (64) $3x - y - z + 2 = 0$
(65) $13x + 3y - 7z - 7 = 0$ (66) $(-1, -1, -1)$ & $5x - 18y + 11z - 2 = 0$

Non-symmetrical form of line :

A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non-symmetrical form.



To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

(i) **Direction ratios:** Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes.
So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by cross-multiplication as $\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$

(ii) **Point on the line** – Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy plane, so it intersect it. Let it intersect xy-plane in $(x_1, y_1, 0)$. Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$.Solving these, we get a point on the line.

Note : If $\ell \neq 0$, then we can take a point on yz-plane as $(0, y_1, z_1)$ and if $m \neq 0$, then we can take a point on xz-plane as $(x_1, 0, z_1)$.

Coplanar lines : Condition of coplanarity if both the lines are in general form Let the lines be

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \ \ \& \ \ \alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Example # 56 : Find the equation of the line of intersection of planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ in the symmetric form.

Solution : Given planes are $4x + 4y - 5z - 12 = 0$ ..... (1)

and $2x - 3y + 4z = 5$ ..... (2)

Let ℓ, m, n be the direction ratios of the line of intersection :

then $4\ell - 4m - 3n = 0$ ..... (3)

and $42\ell - 12m + 13n = 0 \ \therefore \dfrac{\ell}{-8+9} = \dfrac{m}{6-4} = \dfrac{n}{-3+4}$ or, $\dfrac{\ell}{1} = \dfrac{m}{2} = \dfrac{n}{1}$

Hence direction ratios of line of intersection are 1, 2, 1.

Let the line of intersection meet the xy-plane at P $(\alpha, \beta, 0)$.

Then P lies on planes (1) and (2)

$\therefore$ $\alpha - 2\beta = 4$ or, $2\alpha - 3\beta = 5$ ..... (5)

$\alpha = -2$, $\beta = -3$

Hence equation of line of intersection in symmetrical form is $\dfrac{x+2}{1} = \dfrac{y+3}{2} = \dfrac{z}{1}$.

Example # 57 : Find the angle between the lines $x - 3y - 4 = 0$, $4y - z + 5 = 0$ and $x + 3y - 11 = 0$, $2y - z + 6 = 0$.

Solution : Given lines are $\left.\begin{matrix} x - 3y - 4 = 0 \\ 4y - z + 5 = 0 \end{matrix}\right\}$ ..... (1)

and $\left.\begin{matrix} x + 3y - 11 = 0 \\ 2y - z + 6 = 0 \end{matrix}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes

$x - 3y - 4 = 0$ and $4y - z + 5 = 0$

$\therefore$ $\ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3) and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0}$

or, $\frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4}$ = k (let).

Since line (2) is perpendicular to the normals of each of the planes
x + 3y – 11 = 0 and 2y – z + 6 = 0,

∴ $\ell_2 + 3m_2 = 0$ ..... (5) and $2m_2 - n_2 = 0$ ..... (6)

∴ $\ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$ and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

∴ $\frac{\ell_2}{-3} = \frac{m_2}{1} = \frac{n_2}{2}$ = t (let).

If θ be the angle between lines (1) and (2), then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

= (3k) (– 3t) + (k) (t) + (4k) (2t) = – 9kt + kt + 8kt = 0 ∴ θ = 90°.

Example # 58 : Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them.

Solution : Given lines are $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ = r (say)..... (1) and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ = R (say) ..... (2)

If possible, let lines (1) and (2) intersect at P.
Any point on line (1) may be taken as (2r + 3, – 3r – 1, r – 2) = P (let).
Any point on line (2) may be taken as (– 3R + 7, R, 2R – 7) = P (let).

∴ 2r + 3 = – 3R + 7 or, 2r + 3R = 4 ..... (3)
Also – 3r – 1 = R or, – 3r – R = 1 ..... (4)
and r – 2 = 2R – 7 or, r – 2R = – 5. ..... (5)

Solving equations (3) and (4), we get, r = – 1, R = 2
Clearly r = – 1, R = 2 satisfies equation (5).
Hence lines (1) and (2) intersect.

∴ lines (1) and (2) are coplanar.

Equation of the plane containing lines (1) and (2) is $\begin{vmatrix} x-3 & y+1 & z+2 \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0$

or, (x – 3) (– 6 – 1) – (y + 1) (4 + 3) + (z + 2) (2 – 9) = 0
or, – 7 (x – 3) – 7 (y + 1) – 7 (z + 2) = 0 or, x – 3 + y + 1 + z + 2 = 0 or, x + y + z = 0.

Self practice problems:

(67) Find the equation of the line of intersection of the plane
x – y + 2z = 5, 3x + y + z = 6.

(68) Prove that the three planes 2x + y – 4z – 17 = 0, 3x + 2y – 2z – 25 = 0, 2x – 4y + 3z + 25 = 0 intersect at a point and find its co-ordinates.

Answers : (67) $\frac{4x-11}{-33} = \frac{4y+9}{5} = \frac{z}{1}$ (68) (3, 7, –1)

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Position vector, Direction Ratios & Direction cosines

A-1. (i) Let position vectors of points A,B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Point D divides line segment BC internally in the ratio 2 : 1. Find vector $\overrightarrow{AD}$.

(ii) Let ABCD is parallelogram. Position vector of points A,C and D are $\vec{a}$, $\vec{c}$ and $\vec{d}$ respectively. If E divides line segement AB internally in the ratio 3 : 2 then find vector $\overrightarrow{DE}$.

(iii) Let ABCD is trapezium such that $\overrightarrow{AB} = 3\overrightarrow{DC}$. E divides line segement AB internally in the ratio 2 : 1 and F is mid point of DC. If position vector of A,B and C are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively then find vector $\overrightarrow{FE}$.

A-2. In a $\triangle$ABC, $\overrightarrow{AB} = 6\hat{i}+3\hat{j}+3\hat{k}$; $\overrightarrow{AC} = 3\hat{i} - 3\hat{j} + 6\hat{k}$
D and D′ are points trisections of side BC
Find $\overrightarrow{AD}$ and $\overrightarrow{AD'}$.

A-3. If ABCD is a quadrilateral, E and F are the mid-points of AC and BD respectively, then prove that $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD} = 4\overrightarrow{EF}$

A-4.✍ Let ABCD is parallelogram where A = (1,2,4) , B = (8,7,9) and D = (6,1,5) . Find direction cosines of line AC

A-5_. Let one vertex of cuboid whose faces are parallel to coordinate planes is (1, 2, 3). If mid-point of one of the edge in (0, 0,0) then find possible
(i) total surface area of cuboid. (ii) length of body diagonal.

A-6. Find the direction cosines ℓ, m, n of line which are connected by the relations $\ell + m + n = 0$, $2mn + 2m\ell - n\ell = 0$.

Section (B) : Dot Product, Projection and Cross Product

B-1. Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

B-2. If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

B-3. (i) Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where $\vec{a} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}+\hat{k}$.

(ii) Find the projection of the line segment joining (2, – 1, 3) and (4, 2, 5) on a line which makes equal acute angles with co-ordinate axes.

(iii) P and Q are the points (–1, 2, 1) and (4, 3, 5) respectively. Find the projection of PQ on a line which makes angles of 120º and 135º with y and z axes respectively and an acute angle with x-axis.

B-4. Prove that $\left(\dfrac{\vec{a}}{\vec{a}^2}-\dfrac{\vec{b}}{\vec{b}^2}\right)^2 = \left(\dfrac{\vec{a}-\vec{b}}{|\vec{a}|\ |\vec{b}|}\right)^2$

B-5. If $\vec{a}$, $\vec{b}$ are two unit vectors and θ is the angle between them, then show that:

(a) $\sin\frac{\theta}{2} = \frac{1}{2}\left|\vec{a} - \vec{b}\right|$ (b) $\cos\frac{\theta}{2} = \frac{1}{2}\left|\vec{a} + \vec{b}\right|$

B-6. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$, then find the value of $(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b})$.

B-7. For any two vectors $\vec{u}$ & $\vec{v}$, prove that

(a) $(\vec{u}.\vec{v})^2 + |\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2$ & (b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2) = (1-\vec{u}.\vec{v})^2 + |\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

B-8. If the three successive vertices of a parallelogram have the position vectors as, A (– 3, – 2, 0); B (3, – 3, 1) and C (5, 0, 2). Then find

(a) position vector of the fourth vertex D

(b) a vector having the same direction as that of $\overrightarrow{AB}$ but magnitude equal to $\overrightarrow{AC}$

(c) the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

B-9. If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $|\vec{a}| = 5$, $|\vec{b}| = 12$ and $|\vec{c}| = 13$, and $\vec{a}+\vec{b}+\vec{c} = \vec{o}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

B-10. ABCD is a parallelogram in which $\overrightarrow{AB} = 3\hat{i} - 6\hat{j} + 3\hat{k}$ and $\overrightarrow{AD} = 6\hat{i} + 6\hat{j} + 3\hat{k}$. P is a point of AB such that AP : PB = 1 : 2 and Q is a point on BC such that BQ : QC = 2 : 1. Find angle between DQ and PC.

B-11. Prove using vectors : If two medians of a triangle are equal, then it is isosceles.

B-12. (i) Find the angle between the lines whose direction cosines are given by the equations : $3\ell + m + 5n = 0$ and $6mn - 2n\ell + 5\ell m = 0$

(ii) Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 = n^2$.

B-13. Position vectors of A, B, C are given by $\vec{a}$, $\vec{b}$, $\vec{c}$ where $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0$. If $\overrightarrow{AC} = 2\hat{i} - 3\hat{j} + 6\hat{k}$ then find $\overrightarrow{BC}$ if BC = 14.

B-14. A vector $\vec{c}$ is perpendicular to the vectors $2\hat{i} + 3\hat{j} - \hat{k}$, $\hat{i} - 2\hat{j} + 3\hat{k}$ and satisfies the condition $\vec{c}.\left(2\hat{i} - \hat{j} + \hat{k}\right) + 6 = 0$. Find the vector $\vec{c}$

B-15. (a) Show that the perpendicular distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is

$$\frac{\left|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}\right|}{\left|\vec{b} - \vec{a}\right|}.$$

(b) Given a parallelogram ABCD with area 12 sq. units. A straight line is drawn through the mid point M of the side BC and the vertex A which cuts the diagonal BD at a point ' O '. Use vectors to determine the area of the quadrilateral OMCD.

B-16. P, Q are the mid-points of the non-parallel sides BC and AD of a trapezium ABCD. Show that $\Delta APD = \Delta CQB$.

Section (C) : Line

C-1. Find the coordinates of the point when the line through (3, 2, 5) and (–2, 3, –5) crosses the xy plane.

C-2. Find the foot of the perpendicular from (1, 6, 3) on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

C-3. (i) Find the cartesian form of the equation of a line whose vector form is given by .

$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j} - 2\hat{k})$

(ii) Find the vector form of the equation of a line whose cartesian form is given by

$\frac{2x-4}{1} = \frac{3y+6}{2} = \frac{-6z+6}{1}$.

C-4. Find the distance between points of intersection of

Lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ & $\frac{x-4}{5} = \frac{y-1}{2} = z$ and

Lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ & $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

C-5. Show that the foot of the perpendicular from the origin to the join of A(–9, 4, 5) and B (11, 0, –1) is the mid point of AB. Also find distance of point (2, 4, 4) from the line AB.

C-6. Find the equation of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle of $\pi/3$.

C-7. The foot of the perpendicular from (a, b, c) on the line x = y = z is the point (r, r, r), then find the value of r.

C-8. Find the shortest distance between the lines :

$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$

C-9. Let L_1 and L_2 be the lines whose equation are $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ respectively. A and B are two points on L_1 and L_2 respectively such that AB is perpendicular both the lines L_1 and L_2.Find points A, B and hence find shortest distance between lines L_1 and L_2

C-10. If $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$ are two lines, then find the equation of acute angle bisector of two lines.

C-11. The edges of a rectangular parallelopiped are a, b, c; show that the angles between two of the four diagonals are given by $\cos^{-1}\left(\pm\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)\right)$ or $\cos^{-1}\left(\pm\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)\right)$ or $\cos^{-1}\left(\pm\left(\frac{a^2-b^2-c^2}{a^2+b^2+c^2}\right)\right)$.

C-12. Show that equation of angle bisectors of line $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{b} + \mu\vec{a}$ are $\vec{r} = (\vec{a} + \vec{b}) + \gamma\left(|\vec{b}|\vec{a} \pm |\vec{a}|\vec{b}\right)$

C-13. Prove that the shortest distance between the diagonals of a rectangular parallelepiped whose coterminous sides are a, b, c and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

Section (D) : STP, VTP, Vector equation, LI/LD

D-1. Show that $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{c} - \vec{b})\} . \vec{a} = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

D-2. Given unit vectors $\hat{m}$, $\hat{n}$ and $\hat{p}$ such that $\left(\hat{m}\,\hat{\ }\,\hat{n}\right) = \hat{p}\ \hat{\ }\left(\hat{m} \times \hat{n}\right) = \alpha$, then find value of $\left[\hat{n}\ \ \hat{p}\ \ \hat{m}\right]$ in terms of α.

D-3. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to:

D-4. Examine for coplanarity of the following sets of points

(a) $4\hat{i} + 8\hat{j} + 12\hat{k}$, $2\hat{i} + 4\hat{j} + 6\hat{k}$, $3\hat{i} + 5\hat{j} + 4\hat{k}$, $5\hat{i} + 8\hat{j} + 5\hat{k}$.

(b) $3\vec{a} + 2\vec{b} - 5\vec{c}$, $3\vec{a} + 8\vec{b} + 5\vec{c}$, $-3\vec{a} + 2\vec{b} + \vec{c}$, $\vec{a} + 4\vec{b} - 3\vec{c}$. Where $\vec{a}, \vec{b}, \vec{c}$ are noncoplanar

D-5. The vertices of a tetrahedron are P(2, 3, 2), Q(1, 1, 1), R(3, –2, 1) and S (7, 1, 4).

(i) Find the volume of tetrahedron

(ii) Find the shortest distance between the lines PQ & RS.

D-6. Are the following set of vectors linearly independent?

(i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 6\hat{j} + 9\hat{k}$

(ii) $\vec{a} = -2\hat{i} - 4\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$, $\vec{c} = \hat{i} - 4\hat{j} + 3\hat{k}$

D-7. Find value of $x \in R$ for which the vectors $\vec{a} = (1, -2, 3)$, $\vec{b} = (-2, 3, -4)$, $\vec{c} = (1, -1, x)$ form a linearly dependent system.

D-8. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and $\vec{v} . \vec{a} = \vec{v} . \vec{b} = \vec{v} . \vec{c} = 0$, then find value of $\vec{v}$.

D-9. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{d} = 3\hat{i} - \hat{j} - 2\hat{k}$, then

(i) if $\vec{a} \times (\vec{b} \times \vec{c}) = p\vec{a} + q\vec{b} + r\vec{c}$, then find value of p, q and r.

(ii) find the value of $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) . \vec{d}$

D-10. Given that $\vec{x} + \frac{1}{\vec{p}^2}(\vec{p} . \vec{x})\,\vec{p} = \vec{q}$, then show that $\vec{p}.\vec{x} = \frac{1}{2}(\vec{p}.\vec{q})$ and hence find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

D-11. Let there exist a vector $\vec{x}$ satisfying the conditions $\vec{x} \times \vec{a} = \vec{c} \times \vec{d}$ and $\vec{x} + 2\vec{d} = \left(\vec{v} \times \vec{d}\right)$. Find $\vec{x}$ in terms of $\vec{a}$, $\vec{c}$ and $\vec{d}$

Section (E) : Plane

E-1. Find equation of plane

(i) Which passes through (0, 1, 0), (0, 0, 1), (1, 2, 3)

(ii) Which passes through (0, 1, 0) and contains two vectors $\hat{i} + \hat{j} - \hat{k}$ & $2\hat{i} - \hat{j}$.

(iii) Whose normal is $\hat{i} + \hat{j} + \hat{k}$ & which passes through (1, 2, 1).

(iv) Which makes equal intercepts on co-ordinate axis and passes through (1, 2, 3)

E-2. Find the ratio in which the line joining the points (3, 5,–7) and (–2, 1, 8) is divided by the yz-plane. Find also the point of intersection of the plane and the line

E-3. Find the locus of the point whose sum of the square of distances from the planes x + y + z = 0, x – z = 0 and x – 2y + z = 0 is 9

E-4. The foot of the perpendicular drawn from the origin to the plane is (4, –2, –5), then find the vector equation of plane.

E-5. Let P (1, 3, 5) and Q(–2, 1, 4) be two points from which perpendiculars PM and QN are drawn to the x-z plane. Find the angle that the line MN makes with the plane x + y + z = 5.

E-6. The reflection of line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ about the plane x – 2y + z – 6 = 0 is

E-7. Find the equation of image of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane 3x – 3y + 10z = 26.

E-8. Find the angle between the plane passing through points (1, 1, 1), (1, – 1, 1), (–7, –3, –5) & x–z plane.

E-9. Find the equation of the plane containing parallel lines $(x - 4) = \frac{3-y}{4} = \frac{z-2}{5}$ and $(x - 3) = \lambda (y + 2) = \mu z$

E-10. Find the distance of the point (2, 3, 4) from the plane 3x + 2y + 2z + 5 = 0, measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$

E-11. If the acute angle that the vector, $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ makes with the plane of the two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $\hat{i} - \hat{j} + 2\hat{k}$ is $\cot^{-1}\sqrt{2}$ then find the value of $\alpha (\beta + \gamma) - \beta \gamma$

E-12. Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$.

E-13. Find the vector equation of a line passing through the point with position vector $(2\hat{i} - 3\hat{j} - 5\hat{k})$ and perpendicular to the plane $\vec{r}.(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$.
Also, find the point of intersection of this line and the plane.

E-14. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (1, 4, 2) and (2, 3, 5). Also find the coordinates of the foot of the perpendicular and the perpendicular distance of the point (4, 0, 3) from the above found plane.

E-15. Find the equation of the planes passing through points (1, 0, 0) and (0, 1, 0) and making an angle of $0.25\,\pi$ radians with plane x + y – 3 = 0

E-16. Find the distance between the parallel planes $\vec{r}\,.(2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ and $\vec{r}\,.(6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$.

E-17. If the planes x – cy – bz = 0, cx – y + az = 0 and bx + ay – z = 0 pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$ is :

E-18. (i) If $\hat{n}$ is the unit vector normal to a plane and p be the length of the perpendicular from the origin to the plane, find the vector equation of the plane.

(ii) Find the equation of the plane which contains the origin and the line of intersection of the planes $\vec{r}\,.\vec{a} = p$ and $\vec{r}\,.\vec{b} = q$

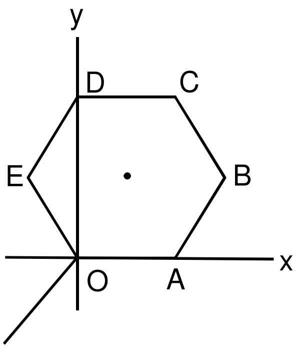
PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Position vector, Direction Ratios & Direction cosines

A-1. If the vector $\vec{b}$ is collinear with the vector $\vec{a} = \left(2\sqrt{2},\ -1,\ 4\right)$ and $|\vec{b}| = 10$, then:

(A) $\vec{a} \pm \vec{b} = 0$ (B) $\vec{a} \pm 2\vec{b} = 0$ (C) $2\vec{a} \pm \vec{b} = 0$ (D) $3\vec{a} \pm \vec{b} = 0$

A-2. OABCDE is a regular hexagon of side 2 units in the XY–plane as shown in figure . O being the origin and OA taken along the X–axis. A point P is taken on a line parallel to Z–axis through the centre of the hexagon at a distance of 3 units from O in the positive Z direction. Then vector $\overline{AP}$ is:



(A) $-\hat{i} + 3\hat{j} + \sqrt{5}\hat{k}$ (B) $\hat{i} - \sqrt{3}\hat{j} + 5\hat{k}$ (C) $-\hat{i} + \sqrt{3}\hat{j} + \sqrt{5}\hat{k}$ (D) $\hat{i} + \sqrt{3}\hat{j} + \sqrt{5}\hat{k}$

A-3. If the sum of the squares of the distances of a point from the three coordinate axes be 36, then its distance from the origin is

(A) 6 (B) $3\sqrt{2}$ (C) $2\sqrt{3}$ (D) $6\sqrt{2}$

A-4. A line makes angles α, β, γ with the coordinate axes. If $\alpha + \beta = 90°$, then γ =

(A) 0 (B) 90º (C) 180º (D) 45°

Section (B) : Dot Product, Projection and Cross Product

B-1. The vector $\frac{1}{2}\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$ is:

(A) a unit vector

(B) makes an angle $\frac{\pi}{3}$ with the vector $2\hat{i} - 4\hat{j} + 3\hat{k}$

(C) parallel to the vector $\hat{i} + \hat{j} - \frac{1}{2}\hat{k}$

(D) perpendicular to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$

B-2. If $|\vec{a}| = 5$, $|\vec{a} - \vec{b}| = 8$ and $|\vec{a} + \vec{b}| = 10$, then $|\vec{b}|$ is equal to :

(A) 1 (B) $\sqrt{57}$ (C) 3 (D) 57

B-3. If $\vec{a}\times\vec{b} = \vec{c}$, $\vec{b}\times\vec{c} = \vec{a}$, then find value of $|3\vec{a}+4\vec{b}+12\vec{c}|$ if $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors of same magnitude.
(A) 11 (B) 12 (C) 13 (D) 14

B-4. Angle between diagonals of a parallelogram whose side are represented by $\vec{a} = 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}-\hat{k}$
(A) $\cos^{-1}\left(\frac{1}{2}\right)$ (B) $\cos^{-1}\left(\frac{3}{5}\right)$ (C) $\sin^{-1}\sqrt{\frac{8}{9}}$ (D) $\tan^{-1}\left(\frac{4}{3}\right)$

B-5. A, B, C & D are four points in a plane with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ respectively such that $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = (\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0$. Then for the triangle ABC, D is its :
(A) incentre (B) circumcentre (C) orthocentre (D) centroid

B-6. Given $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}.\vec{b} = 12$, then find $|\vec{a}\times\vec{b}|$.
(A) 12 (B) 16 (C) 8 (D) 32

B-7. Unit vector perpendicular to the plane of the triangle ABC with position vectors of the vertices A, B, C, is (where Δ is the area of the triangle ABC).
(A) $\frac{(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{\Delta}$ (B) $\frac{(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{2\Delta}$
(C) $\frac{(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{4\Delta}$ (D) $\frac{(\vec{a}\times\vec{b}+\vec{c}\times\vec{b}+\vec{c}\times\vec{a})}{2\Delta}$

B-8. ABC is a triangle where A = (2, 3, 5), B = (–1, 2, 2) and C(λ, 5, μ), if the median through A is equally inclined to the positive axes, then $\lambda + \mu$ is
(A) 7 (B) 6 (C) 15 (D) 9

Section (C) : Line

C-1. If a line has a vector equation $\vec{r} = 2\hat{i} + 6\hat{j} + \lambda(\hat{i} - 3\hat{j})$, then which of the following statement(s) is/are NOT correct?
(A) the line is parallel to $2\hat{i} + 6\hat{j}$ (B) the line passes through the point $3\hat{i} + 3\hat{j}$
(C) the line passes through the point $\hat{i} + 9\hat{j}$ (D) the line is parallel to XY-plane

C-2. Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$. The point of intersection of the lines $\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$ and $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ is :
(A) $-\hat{i} + \hat{j} + 2\hat{k}$ (B) $3\hat{i} - \hat{j} + \hat{k}$ (C) $3\hat{i} + \hat{j} - \hat{k}$ (D) $\hat{i} - \hat{j} - \hat{k}$

C-3. The values of 'λ' for which the two lines $\frac{x-1}{4} = \frac{y-2}{1} = \frac{z}{1}$ & $\frac{x+7}{\lambda} = \frac{y}{\lambda-6} = \frac{z+\lambda}{2}$ are coplaner
(A) 2, 8 (B) 2, –8 (C) 3, 5 (D) 1, 2

C-4. Equation of the angle bisector of the angle between the lines $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1}$ & $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{-1}$ is :

(A) $\frac{x-1}{2} = \frac{y-2}{2}$; $z - 3 = 0$
(B) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$
(C) $x - 1 = 0$; $\frac{y-2}{1} = \frac{z-3}{1}$
(D) $\frac{x-1}{2} = \frac{y-2}{3}$; $z - 3 = 0$

C-5. If the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4}$ and $\frac{x+k}{3} = \frac{y-1}{2} = \frac{z-2}{h}$ are concurrent then

(A) $h = -2, k = -6$ (B) $h = \frac{1}{2}, k = 2$ (C) $h = 6, k = 2$ (D) $h = 2, k = \frac{1}{2}$

C-6. Points X and Y are taken on the sides QR and RS, respectively of a parallelogram PQRS, so that QX = 4XR and RY = 4YS. The line XY cuts the line PR at Z. Find the ratio PZ : ZR.
(A) 4 : 21 (B) 3 : 4 (C) 21 : 4 (D) 4 : 3

Section (D) : STP, VTP, Vector equation, LI/LD

D-1. The value of $\left[(\vec{a}+2\vec{b}-\vec{c})\ (\vec{a}-\vec{b})\ (\vec{a}-\vec{b}-\vec{c})\right]$ is equal to the box product :

(A) $[\vec{a}\,\vec{b}\,\vec{c}]$ (B) $2[\vec{a}\,\vec{b}\,\vec{c}]$ (C) $3[\vec{a}\,\vec{b}\,\vec{c}]$ (D) $4[\vec{a}\,\vec{b}\,\vec{c}]$

D-2. For a non zero vector $\vec{A}$ if the equations $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ and $\vec{A}\times\vec{B} = \vec{A}\times\vec{C}$ hold simultaneously, then :

(A) $\vec{A}$ is perpendicular to $\vec{B}-\vec{C}$
(B) $\vec{A} = \vec{B}$
(C) $\vec{B} = \vec{C}$
(D) $\vec{C} = \vec{A}$

D-3. Let $\vec{a} = x\hat{i}+12\hat{j}-\hat{k}$, $\vec{b} = 2\hat{i}+2x\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}+\hat{k}$. If the ordered set $[\vec{b}\ \vec{c}\ \vec{a}]$ is left handed, then :

(A) $x \in (2, \infty)$ (B) $x \in (-\infty, -3)$ (C) $x \in (-3, 2)$ (D) $x \in \{-3, 2\}$

D-4. If $\vec{a} = i + j - k$, $\vec{b} = i - j + k$, $\vec{c}$ is a unit vector such that $\vec{c}.\vec{a} = 0$, $[\vec{c}\vec{a}\vec{b}] = 0$ then a unit vector $\vec{d}$ both $\vec{a}$ and $\vec{c}$ is perpendicular to

(A) $\frac{1}{\sqrt{6}}(2i - j + k)$ (B) $\frac{1}{\sqrt{2}}(j + k)$ (C) $\frac{1}{\sqrt{2}}(i + j)$ (D) $\frac{1}{\sqrt{2}}(i + k)$

D-5. If $\vec{a} = -i + j + k$ and $\vec{b} = 2i + k$, then the vector $\vec{c}$ satisfying the conditions.
(i) that it is coplanar with $\vec{a}$ and $\vec{b}$
(ii) that its projection on $\vec{b}$ is 0
(A) $-3i + 5j + 6k$ (B) $-3i - 5j + 6k$ (C) $-6i + 5k$ (D) $-i + 2j + 2k$

D-6. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then the vectors $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ are :
(A) non-collinear
(B) linearly independent
(C) perpendicular
(D) parallel

D-7. Vector of length 3 unit which is perpendicular to $\hat{i}+\hat{j}+\hat{k}$ and lies in the plane of $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-3\hat{j}$, is

(A) $\frac{3}{\sqrt{6}}(\hat{i}-2\hat{j}+\hat{k})$ (B) $\frac{3}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$ (C) $\frac{3}{\sqrt{114}}(7\hat{i}+8\hat{j}+\hat{k})$ (D) $\frac{3}{\sqrt{114}}(-7\hat{i}+8\hat{j}-\hat{k})$

D-8. If $\vec{a}, \vec{b}, \vec{c}$ be the unit vectors such that $\vec{b}$ is not parallel to $\vec{c}$ and $\vec{a} \times \left(2\vec{b} \times \vec{c}\right) = \vec{b}$, then the angle that $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are respectively:

(A) $\frac{\pi}{3}$ & $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ & $\frac{2\pi}{3}$ (C) $\frac{\pi}{2}$ & $\frac{2\pi}{3}$ (D) $\frac{\pi}{2}$ & $\frac{\pi}{3}$

D-9. If $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors, then which one of the following set of vectors is linearly dependent?

(A) $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ (B) $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ (C) $\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ (D) $\vec{a}+2\vec{b}+3\vec{c}, \vec{b}-\vec{c}+\vec{a}, \vec{a}+\vec{c}$

D-10. Let $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{r_1} = \vec{a}-\vec{b}+\vec{c}$, $\vec{r_2} = \vec{b}+\vec{c}-\vec{a}$, $\vec{r_3} = \vec{c}+\vec{a}+\vec{b}$, $\vec{r} = 2\vec{a}-3\vec{b}+4\vec{c}$. If $\vec{r} = \lambda_1\vec{r_1}+\lambda_2\vec{r_2}+\lambda_3\vec{r_3}$, then the values of λ_1, λ_2 and λ_3 respectively are

(A) 7, 1, –4 (B) 7 / 2, 1, –1 / 2 (C) 5 / 2, 1, 1/2 (D) –1 / 2, 1, 7 / 2

D-11. Vector $\vec{x}$ satisfying the relation $\vec{A}.\vec{x} = c$ and $\vec{A}\times\vec{x} = \vec{B}$ is

(A) $\frac{c\vec{A}-(\vec{A}\times\vec{B})}{|\vec{A}|}$ (B) $\frac{c\vec{A}-(\vec{A}\times\vec{B})}{|\vec{A}|^2}$ (C) $\frac{c\vec{A}+(\vec{A}\times\vec{B})}{|\vec{A}|^2}$ (D) $\frac{c\vec{A}-2(\vec{A}\times\vec{B})}{|\vec{A}|^2}$

D-12. The value of $\vec{r}$ if exist where $\vec{r} = \vec{a}+\lambda\vec{b}$ and $\vec{r}\times\vec{c} = \vec{d}$ is

(A) $\vec{a} + \left(\frac{\vec{a}.\vec{d}}{\vec{b}.\vec{d}}\right)\vec{b}$ (B) $\vec{a} - \left(\frac{\vec{a}.\vec{d}}{\vec{b}.\vec{d}}\right)\vec{b}$ (C) $\left(\frac{\vec{a}.\vec{d}}{\vec{b}.\vec{d}}\right)\vec{a} - \vec{b}$ (D) $\left(\frac{\vec{a}.\vec{d}}{\vec{b}.\vec{d}}\right)\vec{a} + \vec{b}$

Section (E) : Plane

E-1. The equation of a plane which passes through (2, –3, 1) & is perpendicular to the line joining the points (3, 4, –1) & (2, –1, 5) is given by:

(A) x + 5y – 6z + 19 = 0 (B) x – 5y + 6z – 19 = 0
(C) x + 5y + 6z + 19 = 0 (D) x – 5y – 6z – 19 = 0

E-2. The reflection of the point (2, –1, 3) in the plane 3x – 2y – z = 9 is :

(A) $\left(\frac{26}{7},\frac{15}{7},\frac{17}{7}\right)$ (B) $\left(\frac{26}{7},\frac{-15}{7},\frac{17}{7}\right)$ (C) $\left(\frac{15}{7},\frac{26}{7},\frac{-17}{7}\right)$ (D) $\left(\frac{26}{7},\frac{17}{7},\frac{-15}{7}\right)$

E-3. The distance of the point (– 1, – 5, – 10) from the point of intersection of the line, $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane, x – y + z = 5, is :

(A) 10 (B) 11 (C) 12 (D) 13

E-4. The distance of the point (1, – 2, 3) from the plane x – y + z = 5 measured parallel to the line, $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$, is :

(A) 1 (B) 6/7 (C) 7/6 (D) 1/6

E-5. The distance of the point P(3, 8, 2) from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z + 17 = 0$ is
(A) 2 (B) 3 (C) 5 (D) 7

E-6. If line $\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ is parallel to the plane $\vec{r}.(3\hat{i} - 2\hat{j} - m\hat{k}) = 14$, then the value of m is
(A) 2 (B) – 2
(C) 0 (D) can not be predicted with these informations

E-7. The locus represented by $xy + yz = 0$ is
(A) A pair of perpendicular lines (B) A pair of parallel lines
(C) A pair of parallel planes (D) A pair of perpendicular planes

E-8. The equation of the plane passing through the point (1,–3,–2) and perpendicular to planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$, is
(A) $2x - 4y + 3z - 8 = 0$ (B) $2x - 4y - 3z + 8 = 0$
(C) $2x + 4y + 3z + 8 = 0$ (D) $2x + 4y + 3z - 8 = 0$

E-9. If a plane cuts off intercepts OA = a, OB = b, OC = c from the coordinate axes (where 'O' is the origin), then the area of the triangle ABC is equal to
(A) $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$ (B) $\frac{1}{2}(bc + ca + ab)$
(C) $\frac{1}{2}abc$ (D) $\frac{1}{2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}$

E-10. Given the vertices A (2, 3, 1), B (4, 1, –2), C (6, 3, 7) & D (–5, –4, 8) of a tetrahedron. The length of the altitude drawn from the vertex D is:
(A) 7 (B) 9 (C) 11 (D) 13

PART - III : MATCH THE COLUMN

1. **Column – I** **Column – II**

(A) If the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ constitute the sides of a ΔABC and length of the median bisecting the vector $\vec{c}$ is λ, then λ^2 (p) 2

(B) Let $\vec{p}$ is the position vector of the orthocentre and $\vec{g}$ is the position vector of the centroid of the triangle ABC, where circumcentre is the origin. If $\vec{p} = K\vec{g}$, then K is equal to : (q) 3

(C) Twice of the area of the parallelogram constructed on the vectors $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$, where $\vec{p}$ and $\vec{q}$ are unit vectors containing an angle of 30º, is : (r) 6

(D) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vector such that $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$, $|\vec{w}| = 5$ then $\sqrt{|\vec{u}.\vec{v} + \vec{v}.\vec{w} + \vec{w}.\vec{u}|}$ is (s) 5

2. ✍ Match the following set of lines to the corresponding type :

Column – I		Column – II
(A) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$	& $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}$	(p) parallel but not coincident
(B) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$	& $\frac{x-5}{1}=\frac{y-2}{2}=\frac{z-3}{3}$	(q) intersecting
(C) $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-2}$	& $\frac{x-3}{-1}=\frac{y-4}{-1}=\frac{z-1}{1}$	(r) skew lines
(D) $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{3}$	& $\frac{x}{2}=\frac{y+1}{3}=\frac{z}{1}$	(s) Coincident

3. **Column – I** **Column – II**

(A) The volume of the parallelopiped constructed on the diagonals of the faces of the given rectangular parallelopiped is m times the volume of the given parallelopiped. Then m is equal to — (p) –3

(B) If $\vec{x}$ satisfying the conditions $\vec{b}.\vec{x}=\beta$ & $\vec{b}\times\vec{x}=\vec{a}$ is $\vec{x}=\frac{(\beta^2-12)\vec{b}}{|\vec{b}|^2}+\frac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ then β can be — (q) 2

(C) The points (0, –1, –1), (4, 5, 1), (3, 9, 4) and (–4, 4, k) are coplanar, then k = — (r) 4

(D) ✍ In ΔABC the mid points of the sides AB, BC and CA are respectively (ℓ, 0, 0) (0, m ,0) and (0, 0, n). Then $\frac{AB^2+BC^2+CA^2}{\ell^2+m^2+n^2}$ is equal to — (s) 8

Exercise-2

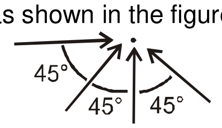
✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1. Let $\vec{a},\vec{b},\vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ to $\vec{c}+\vec{a}$ and $\vec{c}$ to $\vec{a}+\vec{b}$. Then $|\vec{a}+\vec{b}+\vec{c}|$ is equal to :

(A) $2\sqrt{5}$ (B) $2\sqrt{2}$ (C) $10\sqrt{5}$ (D) $5\sqrt{2}$

2. Four coplanar forces are applied at a point O. Each of them is equal to k and the angle between two consecutive forces equals 45º as shown in the figure. Then the resultant has the magnitude equal to :



(A) $k\sqrt{2+2\sqrt{2}}$ (B) $k\sqrt{3+2\sqrt{2}}$ (C) $k\sqrt{4+2\sqrt{2}}$ (D) $k\sqrt{4-2\sqrt{2}}$

3. ✍ Taken on side $\overrightarrow{AC}$ of a triangle ABC, a point M such that $\overrightarrow{AM}=\frac{1}{3}\overrightarrow{AC}$. A point N is taken on the side $\overrightarrow{CB}$ such that $\overrightarrow{BN}=\overrightarrow{CB}$, then for the point of intersection X of $\overrightarrow{AB}$ and $\overrightarrow{MN}$ which of the following holds good?

(A) $\overrightarrow{XB}=\frac{1}{3}\overrightarrow{AB}$ (B) $\overrightarrow{AX}=\frac{1}{3}\overrightarrow{AB}$ (C) $\overrightarrow{XN}=\frac{3}{4}\overrightarrow{MN}$ (D) $\overrightarrow{XM}=3\overrightarrow{XN}$

4. If 3 non zero vectors $\vec{a}, \vec{b}, \vec{c}$ are such that $\vec{a}\times\vec{b} = 2(\vec{a}\times\vec{c})$, $|\vec{a}| = |\vec{c}| = 1$; $|\vec{b}| = 4$ the angle between $\vec{b}$ and $\vec{c}$ is $\cos^{-1}\frac{1}{4}$ then $\vec{b} = \ell\vec{c}+\mu\vec{a}$ where $|\ell| + |\mu|$ is -

(A) 6 (B) 5 (C) 4 (D) 0

5 . If θ is the angle between the vectors $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ and vector $\vec{q} = b\hat{i} + c\hat{j} + a\hat{k}$ then range of θ is

(A) $[0, \pi/3]$ (B) $[\pi/3, 2\pi/3]$ (C) $[0, 2\pi/3]$ (D) $[0, 5\pi/6]$

6. If the unit vectors $\vec{e}_1$ and $\vec{e}_2$ are inclined at an angle 2θ and $|\vec{e}_1 - \vec{e}_2| < 1$, then for $\theta \in [0, \pi]$, θ may lie in the interval :

(A) $\left[0, \frac{\pi}{6}\right)$ (B) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (C) $\left(\frac{5\pi}{6}, \pi\right]$ (D) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

7. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(A) 1/3 (B) 2/3 (C) 4/3 (D) 5/3

9. Given $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $(\vec{a} \wedge \vec{b}) = \frac{\pi}{2}$, $\vec{a}.\vec{c} = 4$, then

(A) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = |\vec{a}|$ (B) $[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{a}|$ (C) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ (D) $[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{a}|^2$

10. The vectors $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+\hat{j}+4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are :

(A) Not coplaner (B) Coplaner but cannot form a triangle
(C) Coplaner but can form a triangle (D) Coplaner & can form a right angled triangle

11. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar unit vectors equally inclined to one another at an acute angle θ. Then $|[\vec{a}\ \vec{b}\ \vec{c}]|$ in terms of θ is equal to :

(A) $(1 + \cos\theta)\sqrt{\cos 2\theta}$ (B) $(1 + \cos\theta)\sqrt{1-2\cos 2\theta}$
(C) $(1 - \cos\theta)\sqrt{1 + 2\cos\theta}$ (D) $(1 - \sin\theta)\sqrt{1 + 2\cos\theta}$

12. Consider a tetrahedron with faces f_1, f_2, f_3, f_4. Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ be the vectors whose magnitudes are respectively equal to the areas of f_1, f_2, f_3, f_4 and whose directions are perpendicular to these faces in the outward direction. Then,

(A) $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4 = \vec{0}$ (B) $\vec{a}_1 + \vec{a}_3 = \vec{a}_2 + \vec{a}_4$
(C) $\vec{a}_1 + \vec{a}_2 = \vec{a}_3 + \vec{a}_4$ (D) $\vec{a}_1 + \vec{a}_2 - \vec{a}_3 + \vec{a}_4 = \vec{0}$

13. Let $\vec{r}$ be a vector perpendicular to $\vec{a} + \vec{b} + \vec{c}$, where $[\vec{a}\ \vec{b}\ \vec{c}] = 2$. If $\vec{r} = \ell(\vec{b} \times \vec{c}) + m(\vec{c} \times \vec{a}) + n(\vec{a} \times \vec{b})$, then $(\ell + m + n)$ is equal to

(A) 2 (B) 1 (C) 0 (D) –1

14. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar non-zero vectors and $\vec{r}$ is any vector in space, then $(\vec{a}\times\vec{b}) \times (\vec{r}\times\vec{c}) + (\vec{b}\times\vec{c}) \times (\vec{r}\times\vec{a}) + (\vec{c}\times\vec{a}) \times (\vec{r}\times\vec{b})$ is equal to

(A) $2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (B) $3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (C) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (D) $4[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$

15. If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a} \parallel (\vec{b} \times \vec{c})$, then $(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c})$ is equal to

(A) $\vec{a}^2(\vec{b}.\vec{c})$ (B) $\vec{b}^2(\vec{a}.\vec{c})$ (C) $\vec{c}^2(\vec{a}.\vec{b})$ (D) $-\vec{a}^2(\vec{b}.\vec{c})$

16. Let $2\vec{c}+\vec{b}$ = $4\vec{a}+3\vec{d}$. If $\left[\vec{d}\ \vec{c}\ \vec{a}\right]$ and $\left[\vec{a}\ \vec{b}\ \vec{d}\right]$ are natural numbers with H.C.F. equal to 1 then how many statement are true among below six statement.

(i) $\left[\vec{a}\ \vec{b}\ \vec{d}\right]=2$ (ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=3$

(iii) $\left[\vec{d}\ \vec{c}\ \vec{b}\right]=4$ (iv) $\left[\vec{d}\ \vec{c}\ \vec{a}\right]=1$

(v) $\left(\vec{a}\times\vec{b}\right) \times \left(\vec{c}\times\vec{d}\right) = 2\vec{c} - 3\vec{d}$ (vi) $\left(\vec{a}\times\vec{b}\right) \times \left(\vec{c}\times\vec{d}\right) = 4\vec{a}-\vec{b}$

(A) 2 (B) 4 (C) 6 (D) 0

17. Let $\vec{u}$ and $\vec{v}$ are unit vectors and $\vec{w}$ is a vector such that $(\vec{u}\times\vec{v})+\vec{u}$ = $\vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$ then the value of $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]$ is equal to

(A) 1 (B) 2 (C) 0 (D) –1

18. Find the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0, x + z = 0, x + y = 0, x + y + z = \sqrt{3}\,a$.

(A) a (B) 2a (C) $a/\sqrt{2}$ (D) $\sqrt{2}$ a.

19. A plane meets the coordinate axes in A, B, C and (α, β, γ) is the centroid of the triangle ABC, then the equation of the plane is

(A) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$ (B) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ (C) $\frac{3x}{\alpha}+\frac{3y}{\beta}+\frac{3z}{\gamma}=1$ (D) $\alpha x + \beta y + \gamma z = 1$

20. Equation of plane which passes through the point of intersection of lines $\frac{x-1}{3}$ = $\frac{y-2}{1}$ = $\frac{z-3}{2}$ and $\frac{x-3}{1}$ = $\frac{y-1}{2}$ = $\frac{z-2}{3}$ and at greatest distance from the point (0, 0, 0) is:

(A) $4x + 3y + 5z = 25$ (B) $4x + 3y + 5z = 50$ (C) $3x + 4y + 5z = 49$ (D) $x + 7y - 5z = 2$

21. The non zero value of 'a' for which the lines $2x - y + 3z + 4 = 0 = ax + y - z + 2$ and $x - 3y + z = 0 = x + 2y + z + 1$ are co-planar is :

(A) – 2 (B) 4 (C) 6 (D) 0

22. A line having direction ratios 3, 4, 5 cuts 2 planes $2x - 3y + 6z - 12 = 0$ and $2x - 3y + 6z + 2 = 0$ at point P & Q, then find length of PQ

(A) $\frac{35\sqrt{2}}{12}$ (B) $\frac{35\sqrt{2}}{24}$ (C) $\frac{35\sqrt{2}}{6}$ (D) $\frac{35\sqrt{2}}{8}$

23. A line L_1 having dierection ratios 1, 0, 1 lies on xz plane. Now this xz plane is rotated about z-axis by an angle of 90°. Now the new position of L_1 is L_2. The angle between L_1 & L_2 is :

(A) 30° (B) 60° (C) 90° (D) 45°

24. Focii of ellipse lie on plane $2x + 3y + 6z = 0$ are (54, 18, –27) and (–54, –18, 27). If eccentricity of ellipse is $\frac{3}{5}$ then find equation of directrices .

(A) $\frac{x-150}{3} = \frac{y-50}{-6} = \frac{z+75}{2}, \frac{x+150}{3} = \frac{y+50}{-6} = \frac{z-75}{2}$

(B) $\frac{x-150}{-3} = \frac{y+50}{-6} = \frac{z+75}{2}, \frac{x+150}{-3} = \frac{y-50}{-6} = \frac{z-75}{2}$

(C) $\frac{x-150}{3} = \frac{y+50}{-6} = \frac{z+75}{2}, \frac{x+150}{3} = \frac{y-50}{-6} = \frac{z-75}{2}$

(D) $\frac{x-150}{3} = \frac{y-50}{-6} = \frac{z-75}{-2}, \frac{x+150}{3} = \frac{y+50}{-6} = \frac{z+75}{-2}$

PART-II: NUMERICAL VALUE QUESTIONS

INSTRUCTION :

- The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.
- If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

1. Given $f^2(x) + g^2(x) + h^2(x) \le 9$ and $U(x) = 3f(x) + 4g(x) + 10h(x)$, where f(x), g(x) and h(x) are continuous $\forall\, x \in R$ then maximum value of U(x) is.

2. If in a plane $A_1, A_2, A_3, \ldots\ldots A_{20}$ are the vertices of a regular polygon having 20 sides and O is its centre and $\sum_{i=1}^{19}(\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = \lambda\,(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$ then $|\lambda|$ is

3. In an equilateral ΔABC find the value of $\dfrac{|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2}{R^2}$ where P is any arbitrary point lying on its circumcircle, is

4. Let $\vec{A} = 2\hat{i} + \hat{k}$, $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$. If a vector $\vec{R} = \alpha\hat{i} - \beta\hat{j} + \gamma\hat{k}$ satisfies $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$ then $\alpha^2 + \beta^2 + \gamma^2$ is

5. A line $\dfrac{x+2}{1} = \dfrac{y-3}{2} = \dfrac{z-k}{3}$ cuts the y-z plane and the x-y plane at A and B respectively. If $\angle AOB = \dfrac{\pi}{2}$, then k, where O is the origin, is

6. Given four non zero vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar but not collinear pair by pair and vector $\vec{d}$ is not coplanar with vectors $\vec{a}, \vec{b}$ and $\vec{c}$ and $(\widehat{\vec{a}\,\vec{b}}) = (\widehat{\vec{b}\,\vec{c}}) = \dfrac{\pi}{3}$, $(\widehat{\vec{d}\,\vec{a}}) = \alpha$ and $(\widehat{\vec{d}\,\vec{b}}) = \beta$, if $(\widehat{\vec{d}\,\vec{c}}) = \cos^{-1}(m\cos\beta + n\cos\alpha)$ then m – n is :

7. If the circumcentre of the tetrahedron OABC is given by $\dfrac{\vec{a}^2\left(\vec{b} \times \vec{c}\right) + \vec{b}^2\left(\vec{c} \times \vec{a}\right) + \vec{c}^2\left(\vec{a} \times \vec{b}\right)}{\alpha}$, where $\vec{a}, \vec{b}$ & $\vec{c}$ are the position vectors of the points A, B, C respectively relative to the origin 'O' such that $[\vec{a}\ \vec{b}\ \vec{c}] = 36$ then α is

8. Given three point on x – y plane as O(0, 0), A(1, 0) & B(–1, 0). Point P moving on the given plane satisfying the condition $(\overrightarrow{PA} \cdot \overrightarrow{PB}) + 3(\overrightarrow{OA} \cdot \overrightarrow{OB}) = 0$
If the maximum & minimum values of $|\overrightarrow{PA}|\,|\overrightarrow{PB}|$ is M & m respectively then the value of $M^2 + m^2$ is

9. If the volume of tetrahedron formed by planes whose equations are y + z = 0, z + x = 0, x + y = 0 and x + y + z = 1 is t cubic unit, then the value of t is

10. If $\vec{r}$ represents the position vector of point R in which the line AB cuts the plane CDE, where position vectors of points A, B, C, D, E are respectively $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{c} = -4\hat{j} + 4\hat{k}$, $\vec{d} = 2\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{e} = 4\hat{i} + \hat{j} + 2\hat{k}$, then $|\vec{r}|$ is :

11. Line L_1 is parallel to vector $\vec{\alpha} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and passes through a point A(7, 6, 2) and line L_2 is parallel to a vector $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ and passes through a point B(5, 3, 4). Now a line L_3 parallel to a vector $\vec{r} = 2\hat{i} - 2\hat{j} - \hat{k}$ intersects the lines L_1 and L_2 at points C and D respectively, then $|4\overrightarrow{CD}|$ is equal to :

12. L is the equation of the straight line which passes through the point (2, –1, –1); is parallel to the plane 4x + y + z + 2 = 0 and is perpendicular to the line of intersection of the planes 2x + y = 0 = x – y + z. If the point (3, α, β) lies on line L, then absolute value of $\frac{\alpha}{\beta}$ is

13. The lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ and 3x – 2y + z + 5 = 0 = 4x + 3y – 4z – 3k are coplanar, then the value of k is

14. About the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ the plane 3x + 4y + 6z + 7 = 0 is rotated till the plane passes through the origin. Now 4x + αy + βz = 0 is the equation of plane in new position. The value of $\alpha^2 + \beta^2$ is

15. The value of $\tan^3\theta$, where θ is the acute angle between the plane faces of a regular tetrahedron, is

16. R and r are the circum–radius and in–radius of a regular tetrahedron respectively in terms of the length k of each edge. If $R^2 + r^2 = \lambda k^2$ then value of λ is

17. A line L on the plane 2x + y – 3z + 5 = 0 is at a distance 3 unit from the point P(1, 2, 3). A spider starts moving from point A and after moving 4 units along the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-3}$ it reaches to point P. and from P it jumps to line L along the shortest distance and then moves 12 units along the line L to reach at point B. The distance between points A and B is

18. The length of edge of a regular tetrahedron D-ABC is 'a'. Point E & F are taken on the edges AD and BD respectively. Such that E divide $\overrightarrow{DA}$ and F divide $\overrightarrow{BD}$ in the ratio 2 : 1 each. The area of ΔCEF is equal to $\frac{\sqrt{3}}{k}a^2$, then value of k is :

19. If 'd' be the shortest distance between the lines $\frac{y}{b} + \frac{z}{c} = 1$; x = 0 and $\frac{x}{a} - \frac{z}{c} = 1$; y = 0 and if $\frac{\lambda}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ then λ is

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. A vector $\vec{a}$ has components 2p and 1 with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counterclockwise sense. If with respect to the new system, $\vec{a}$ has components p+1 and 1, then

(A) $p = -\frac{1}{3}$ (B) p = 1 (C) p = – 1 (D) $p = \frac{1}{3}$

2. If $\vec{z}_1 = a\hat{i} + b\hat{j}$ and $\vec{z}_2 = c\hat{i} + d\hat{j}$ are two vectors in $\hat{i}$ and $\hat{j}$ system, where $|\vec{z}_1| = |\vec{z}_2|$ = r and $\vec{z}_1 \cdot \vec{z}_2 = 0$, then $\vec{w}_1 = a\hat{i} + c\hat{j}$ and $\vec{w}_2 = b\hat{i} + d\hat{j}$ satisfy :

(A) $|\vec{w}_1| = r$ (B) $|\vec{w}_2| = r$ (C) $\vec{w}_1 . \vec{w}_2 = 0$ (D) $|\vec{w}_1| \neq |\vec{w}_2|$

3. If a, b, c, x, y, z ∈ R such that ax + by + cz = 2, then which of the following is always true
(A) $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq 4$
(B) $(x^2 + b^2 + z^2)(a^2 + y^2 + c^2) \geq 4$
(C) $(a^2 + y^2 + z^2)(x^2 + b^2 + c^2) \geq 4$
(D) $(a^2 + b^2 + z^2)(x^2 + y^2 + c^2) \geq 4$

4. The direction cosines of the lines bisecting the angle between the lines whose direction cosines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and the angle between these lines is θ, are
(A) $\frac{\ell_1 - \ell_2}{\cos\frac{\theta}{2}}, \frac{m_1 - m_2}{\cos\frac{\theta}{2}}, \frac{n_1 - n_2}{\cos\frac{\theta}{2}}$
(B) $\frac{\ell_1 + \ell_2}{2\cos\frac{\theta}{2}}, \frac{m_1 + m_2}{2\cos\frac{\theta}{2}}, \frac{n_1 + n_2}{2\cos\frac{\theta}{2}}$
(C) $\frac{\ell_1 + \ell_2}{2\sin\frac{\theta}{2}}, \frac{m_1 + m_2}{2\sin\frac{\theta}{2}}, \frac{n_1 + n_2}{2\sin\frac{\theta}{2}}$
(D) $\frac{\ell_1 - \ell_2}{2\sin\frac{\theta}{2}}, \frac{m_1 - m_2}{2\sin\frac{\theta}{2}}, \frac{n_1 - n_2}{2\sin\frac{\theta}{2}}$

5. The value(s) of $\alpha \in [0, 2\pi]$ for which vector $\vec{a} = \hat{i} + 3\hat{j} + (\sin 2\alpha)\ \hat{k}$ makes an obtuse angle with the z-axis and the vectors $\vec{b} = (\tan\alpha)\ \hat{i} - \hat{j} + 2\sqrt{\sin\frac{\alpha}{2}}\ \hat{k}$ and $\vec{c} = (\tan\alpha)\,\hat{i} + (\tan\alpha)\ \hat{j} - 3\sqrt{\operatorname{cosec}\frac{\alpha}{2}}\ \hat{k}$ are orthogonal, is/are
(A) $\tan^{-1} 3$
(B) $\pi - \tan^{-1} 2$
(C) $\pi + \tan^{-1} 3$
(D) $2\pi - \tan^{-1} 2$

6. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle of $\cos^{-1}\frac{11}{14}$ and doubled in magnitude, then it becomes $4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$. The value of 'x' **CANNOT** be :
(A) $-\frac{2}{3}$
(B) $\frac{2}{3}$
(C) $-\frac{20}{17}$
(D) 2

7. The vertices of a triangle are A (1, 1, 2), B(4, 3, 1) and C(2, 3, 5). A vector representing the bisector of the angle A is :
(A) $2\hat{i} - 4\hat{k}$
(B) $-2\hat{i} + 4\hat{k}$
(C) $-2\hat{i} - 2\hat{j} - \hat{k}$
(D) $2\hat{i} + 2\hat{j} + \hat{k}$

8. The vector $\vec{c}$, parallel to the internal bisector of the angle between the vectors $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$ with $|\ \vec{c}\ | = 5\sqrt{6}$, is :
(A) $\frac{5}{3}\left(\hat{i} - 7\hat{j} + 2\hat{k}\right)$
(B) $\frac{5}{3}\left(\hat{i} + 7\hat{j} - 2\hat{k}\right)$
(C) $\frac{5}{3}\left(-\hat{i} + 7\hat{j} - 2\hat{k}\right)$
(D) $\frac{5}{3}\left(-\hat{i} - 7\hat{j} + 2\hat{k}\right)$

9. A line passes through a point A with position vector $3\hat{i} + \hat{j} - \hat{k}$ and is parallel to the vector $2\hat{i} - \hat{j} + 2\hat{k}$. If P is a point on this line such that AP = 15 units, then the position vector of the point P is/are
(A) $13\hat{i} + 4\hat{j} - 9\hat{k}$
(B) $13\hat{i} - 4\hat{j} + 9\hat{k}$
(C) $7\hat{i} - 6\hat{j} + 11\hat{k}$
(D) $-7\hat{i} + 6\hat{j} - 11\hat{k}$

10. Acute angle between the lines $\frac{x-1}{\ell} = \frac{y+1}{m} = \frac{z}{n}$ and $\frac{x+1}{m} = \frac{y-3}{n} = \frac{z-1}{\ell}$ where $\ell > m > n$, and ℓ, m, n are the roots of the cubic equation $x^3 + x^2 - 4x = 4$ is equal to :
(A) $\cos^{-1}\frac{3}{\sqrt{13}}$
(B) $\sin^{-1}\frac{\sqrt{65}}{9}$
(C) $2\cos^{-1}\sqrt{\frac{13}{18}}$
(D) $\tan^{-1}\frac{2}{3}$

11. The line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the curve $xy = c^2$, z = 0 if c is equal to :
(A) –1
(B) $-\sqrt{5}$
(C) $\sqrt{5}$
(D) 1

12. Three distinct lines $\frac{x-1}{3}=\frac{y-2}{2}=\frac{z-3}{1}$, $\frac{x-1}{5}=\frac{2y-4}{3}=\frac{3z-9}{1}$, $\frac{x-\lambda^2}{3}=\frac{y-2}{2}=\frac{z-3}{\lambda}$ are concurrent the value of λ may be :

(A) 1 (B) –1 (C) 2 (D) –2

13. The line $\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$ is the hypotenuse of an isosceles right angle triangle whose opposite vertex is (7, 2, 4) Then the equation of remaining sides is/are -

(A) $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$ (B) $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

(C) $\frac{x+7}{3}=\frac{y+2}{6}=\frac{z+4}{2}$ (D) $\frac{x+7}{2}=\frac{y+2}{-3}=\frac{z+4}{6}$

14. Two lines are

$L_1 : \frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{2}$; $L_2 : \frac{x-1}{2}=\frac{y-1}{1}=\frac{z-1}{2}$

Equation of line passing through (2, 1, 3) and equally inclined to L_1 & L_2 is/are

(A) $\frac{x-2}{2}=\frac{y-1}{2}=\frac{z-3}{-3}$ (B) $\frac{x-3}{1}=\frac{y-2}{1}=\frac{z-5}{2}$

(C) $\frac{x-2}{1}=\frac{y-1}{1}=\frac{z-3}{3}$ (D) $\frac{x}{2}=\frac{y+1}{2}=\frac{z-6}{-3}$

15. Which of the followings is/are correct :

(A) The angle between the two straight lines $\vec{r} = 3\hat{i} - 2\hat{j} + 4\hat{k} + \lambda(-2\hat{i} + \hat{j} + 2\hat{k})$ and

$\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k} + \mu(3\hat{i} - 2\hat{j} + 6\hat{k})$ is $\cos^{-1}\left(\frac{4}{21}\right)$

(B) $(\vec{r}.\hat{i})\ (\hat{i}\times\vec{r})+(\vec{r}.\hat{j})\ (\hat{j}\times\vec{r})+(\vec{r}.\hat{k})\ (\hat{k}\times\vec{r})=\vec{0}$.

(C) The force determined by the vector $\vec{r}$ = (1, –8, –7) is resolved along three mutually perpendicular directions, one of which is in the direction of the vector $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$. Then the vector component of the force $\vec{r}$ in the direction of the vector $\vec{a}$ is $-\frac{7}{3}(2\hat{i}+2\hat{j}+\hat{k})$

(D) The cosine of the acute angle between any two diagonals of a cube is $\frac{1}{3}$.

16. If the distance between points (α, 5α, 10α) from the point of intersection of the line.
$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 12\hat{k})$ and plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$ is 13 units, then value of α may be

(A) 1 (B) – 1 (C) 4 (D) $\frac{80}{63}$

17. A vector $\vec{v} = \lambda\left(a\hat{j} + b\hat{k}\right)$ is coplanar with the vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 2\hat{j} + \hat{k}$ and is orthogonal to the vector $-2\hat{i} + \hat{j} + \hat{k}$. It is given that the length of projection of $\vec{v}$ along the vector $\hat{i} - \hat{j} + \hat{k}$ is equal to $6\sqrt{3}$. Then the value of $\lambda^2 ab$ may be

(A) 81 (B) 9 (C) –9 (D) –81

18. $\hat{a}$ and $\hat{b}$ are two given unit vectors at right angle. The unit vector equally inclined with $\hat{a}$, $\hat{b}$ and $\hat{a} \times \hat{b}$ will be:

(A) $-\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}+\hat{a}\times\hat{b}\right)$ (B) $\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}+\hat{a}\times\hat{b}\right)$ (C) $\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}-\hat{a}\times\hat{b}\right)$ (D) $-\frac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}-\hat{a}\times\hat{b}\right)$

19. Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose length of projection on $\vec{a}$ is of $\sqrt{\frac{2}{3}}$, is

(A) 2i + 3j – 3k (B) 2i + 3j + 3k (C) – 2i – j + 5k (D) i – 5j + 3k

20. The position vectors of the angular points of a tetrahedron are $A(3\hat{i}-2\hat{j}+\hat{k})$, $B(3\hat{i}+\hat{j}+5\hat{k})$, $C(4\hat{i}+3\hat{k})$ and $D(\hat{i})$. Then the acute angle between the lateral face ADC and the base face ABC is :

(A) $\tan^{-1}\frac{5}{2}$ (B) $\tan^{-1}\frac{2}{5}$ (C) $\cot^{-1}\frac{5}{2}$ (D) $\cot^{-1}\frac{2}{5}$

21. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \lambda$ and $\vec{a} \,.\, \vec{c} = \frac{\sqrt{3}}{2}$, then

(A) $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if $\lambda = 1$
(B) Angle between $\vec{b}$ and $\vec{d}$ is 30° if $\lambda = -1$
(C) angle between $\vec{b}$ and $\vec{d}$ is 150° if $\lambda = -1$
(D) If $\lambda = 1$ then angle between $\vec{b}$ and $\vec{c}$ is 60°

22. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. If the position vectors of the vertices of the base ABC are A(1, 0, 1), B(2,0, 0) and C(0, 1, 0), then position vectors of the vertex A_1 can be:

(A) (2, 2, 2) (B) (0, 2, 0) (C) (0, –2, 2) (D) (0, –2, 0)

23. The coplanar points A , B , C , D are (2 – x , 2 , 2) , (2 , 2 – y , 2) , (2 , 2 , 2 – z) and (1 , 1 , 1) respectively, then

(A) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$
(B) $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=2$
(C) $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=1$
(D) $\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}+2=0$

24. Which of the following statement(s) is/are correct :

(A) If $\vec{a},\vec{b},\vec{c}$ are non–coplanar and $\vec{d}$ is any vector, then

$$[\vec{d}\ \vec{b}\ \vec{c}]\ \vec{a}+[\vec{d}\ \vec{c}\ \vec{a}]\ \vec{b}+[\vec{d}\ \vec{a}\ \vec{b}]\ \vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\ \vec{d}=\vec{0}$$

(B) If I is incentre of Δ ABC then $|\overline{BC}|\,\overline{IA}+|\overline{CA}|\,\overline{IB}+|\overline{AB}|\,\overline{IC}=\vec{0}$

(C) Any vector in three dimension can be written as linear combination of three non–coplanar vectors.

(D) In a triangle, if position vector of vertices are $\vec{a},\vec{b},\vec{c}$, then position vector of incentre is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

25. Let $\vec{V} = 2\hat{i}+\hat{j}-\hat{k}$ & $\vec{W}=\hat{i}+3\hat{k}$. If $\vec{U}$ is a unit vector, then the value of the scalar triple product $\left[\vec{U}\ \vec{V}\ \vec{W}\right]$ may be :

(A) $-\sqrt{59}$ (B) $\sqrt{10}+\sqrt{6}$ (C) $\sqrt{59}$ (D) $\sqrt{60}$

26. If $\vec{A}+\vec{B} = \vec{a}$, $\vec{A}.\vec{a} = 1$ and $\vec{A}\times\vec{B} = \vec{b}$, then

(A) $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ (B) $\vec{B} = \dfrac{\vec{a}\times\vec{b}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$

(C) $\vec{A} = \dfrac{\vec{b}\times\vec{a}+\vec{a}}{|\vec{a}|^2}$ (D) $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\left(|\vec{a}|^2-1\right)}{|\vec{a}|^2}$

27. The line $\dfrac{x-4}{k} = \dfrac{y-2}{1} = \dfrac{z-k^2}{2}$ lies in the plane $2x - 4y + z = 1$. Then the value of k cannot be :

(A) 1 (B) – 1 (C) 2 (D) –2

28. Equation of the plane passing through $A(x_1, y_1, z_1)$ and containing the line $\dfrac{x-x_2}{d_1} = \dfrac{y-y_2}{d_2} = \dfrac{z-z_2}{d_3}$ is

(A) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$ (B) $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

(C) $\begin{vmatrix} x-d_1 & y-d_2 & z-d_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$ (D) $\begin{vmatrix} x & y & z \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

29. A line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ intersects the plane $x - y + 2z + 2 = 0$ at point A. The equation of the straight line passing through A lying in the given plane and at minimum inclination with the given line is/are

(A) $\dfrac{x+1}{1} = \dfrac{y+1}{5} = \dfrac{z+1}{2}$ (B) $5x - y + 4 = 0 = 2y - 5z - 3$

(C) $5x + y - 5z + 1 = 0 = 2y - 5z - 3$ (D) $\dfrac{x+2}{1} = \dfrac{y+6}{5} = \dfrac{z+3}{2}$

30. If the π-plane $7x + (\alpha + 4)y + 4z - r = 0$ passing through the points of intersection of the planes $2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$ and is perpendicular to the plane $3x - y - 2z = 4$ and $\left(\dfrac{12}{\beta}, \dfrac{-78}{\beta}, \dfrac{57}{\beta}\right)$ is image of point (1, 1, 1) in π–plane, then

(A) $\alpha = 9$ (B) $\beta = -117$ (C) $\alpha = -9$ (D) $\beta = 117$

31. The planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$ and $8x - 31y - 33z = 0$

(A) pass through origin (B) intersect in a common line

(C) form a triangular prism (D) pass through infinite the many points

32. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are the position vectors of the points A, B, C and D respectively in three dimensional space no three of A, B, C, D are collinear and satisfy the relation $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d} = \vec{0}$, then :

(A) A, B, C and D are coplanar

(B) The line joining the points B and D divides the line joining the point A and C in the ratio 2 : 1.

(C) The line joining the points A and C divides the line joining the points B and D in the ratio 1 : 1.

(D) The four vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are linearly dependents.

PART - IV : COMPREHENSION

Comprehension # 1

In a parallelogram OABC, vectors $\vec{a}, \vec{b}, \vec{c}$ are respectively the position vectors of vertices A, B, C with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of 2 : 1 internally.

Also, the line segment AE intersect the line bisecting the angle O internally in point P. If CP, when extended meets AB in point F. Then

1. The position vector of point P, is

(A) $\frac{3|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{c}}{|\vec{c}|}\right\}$ (B) $\frac{|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{c}}{|\vec{c}|}\right\}$

(C) $\frac{2|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{c}}{|\vec{c}|}\right\}$ (D) $\frac{3|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\frac{\vec{a}}{|\vec{a}|}-\frac{\vec{c}}{|\vec{c}|}\right\}$

2. The position vector of point F, is

(A) $\vec{a}+\frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (B) $\vec{a}+\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (C) $\vec{a}+\frac{2|\vec{a}|}{|\vec{c}|}\vec{c}$ (D) $\vec{a}-\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$

3. The vector $\overrightarrow{AF}$, is given by

(A) $-\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (B) $\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (C) $\frac{2|\vec{a}|}{|\vec{c}|}\vec{c}$ (D) $\frac{1}{3}\frac{|\vec{a}|}{|\vec{c}|}\vec{c}$

Comprehension # 2

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be two planes, where $d_1, d_2 > 0$. Then origin lies in acute angle if $a_1a_2 + b_1b_2 + c_1c_2 < 0$ and origin lies in obtuse angle if $a_1a_2 + b_1b_2 + c_1c_2 > 0$.
Further point (x_1, y_1, z_1) and origin both lie either in acute angle or in obtuse angle ,
if $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) > 0$, one of (x_1, y_1, z_1) and origin lie in acute angle and the other in obtuse angle, if $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) < 0$

4. Given the planes $2x + 3y – 4z + 7 = 0$ and $x – 2y + 3z – 5 = 0$, if a point P is (1, – 2, 3) and O is origin, then
(A) O and P both lie in acute angle between the planes
(B) O and P both lie in obtuse angle between the planes
(C) O lies in acute angle, P lies in obtuse angle.
(D) O lies in obtue angle, P lies in acute angle.

5. Given the planes $x + 2y – 3z + 5 = 0$ and $2x + y + 3z + 1 = 0$. If a point P is (2, –1, 2) and O is origin, then
(A) O and P both lie in acute angle between the planes
(B) O and P both lie in obtuse angle between the planes
(C) O lies in acute angle, P lies in obtuse angle. (D) O lies in obtue angle, P lies in acute angle.

6. Given the planes $x + 2y – 3z + 2 = 0$ and $x – 2y + 3z + 7 = 0$, if the point P is (1, 2, 2) and O is origin, then
(A) O and P both lie in acute angle between the planes
(B) O and P both lie in obtuse angle between the planes
(C) O lies in acute angle, P lies in obtuse angle. (D) O lies in obtue angle, P lies in acute angle.

Comprehension # 3

If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}',\vec{b}',\vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$, then the two systems are called Reciprocal System of vectors and $\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

w7. Find the value of $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'$.

(A) $\vec{0}$ (B) $\vec{a}+\vec{b}+\vec{c}$ (C) $\vec{a}-\vec{b}+\vec{c}$ (D) $\vec{a}+\vec{b}-\vec{c}$

8. Find value of λ such that $\vec{a}'\times\vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \lambda\, \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$.

(A) – 1 (B) 1 (C) 2 (D) – 2

9. If $[(a' \times b') \times (b' \times c') \quad (b' \times c') \times (c' \times a') \quad (c' \times a') \times (a' \times b')] = [abc]^n$, then find n.

(A) n = – 4 (B) n = 4 (C) n = – 3 (D) n = 3

Comprehension # 4

The vertices of square pyramid are A(0, 0, 0), B(4, 0, 0), C(4, 0, 4), D(0, 0, 4) and E(2, 6, 6)

10. Volume of the pyramid is :

(A) 32 (B) 16 (C) 8 (D) 4

11. Centroids of triangular faces of square pyramid are

(A) Non-coplanar (B) Coplanar but the plane is not parallel to base plane

(C) Coplanar & plane is parallel to base plane (D) Co-linear

12. The distance of the plane EBC from ortho-centre of ΔABD is :

(A) 2 (B) 5 (C) $\dfrac{12}{\sqrt{10}}$ (D) $\sqrt{10}$

Comprehension # 5

General equation of a sphere is given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, where $(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.

Let P be a any plane and F is the foot of perpendicular from centre(C) of the sphere to this plane.

If CF > $\sqrt{u^2+v^2+w^2-d}$ then plane P neither touches nor cuts the sphere.

If CF = $\sqrt{u^2+v^2+w^2-d}$ then plane P touches the sphere.

If CF < $\sqrt{u^2+v^2+w^2-d}$ then intersection of plane P and sphere is a circle with

radius = $\sqrt{u^2+v^2+w^2-d-(CF)^2}$

13. Find the equation of the sphere having centre at (1, 2, 3) and touching the plane $x + 2y + 3z = 0$.

(A) $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$ (B) $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$

(C) $x^2 + y^2 + z^2 - 2x - 4y + 6z = 0$ (D) $x^2 + y^2 + z^2 + 2x - 4y - 6z = 0$

14. A variable plane passes through a fixed point (1, 2, 3). The locus of the foot of the perpendicular drawn from origin to this plane is:

(A) $x^2 + y^2 + z^2 - x - 2y - 3z = 0$ (B) $x^2 + 2y^2 + 3z^2 - x - 2y - 3z = 0$

(C) $x^2 + 4y^2 + 9z^2 + x + 2y + 3 = 0$ (D) $x^2 + y^2 + z^2 + x + 2y + 3z = 0$

15. Find the length of the chord intercepted on the line $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$ by the sphere

$x^2 + y^2 + z^2 - 2x - 2y - \dfrac{22}{3}z = 0$.

(A) $\sqrt{56}$ (B) $\sqrt{54}$ (C) 9 (D) 6

Exercise-3

* **Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a **IIT-JEE-2010, Paper-1, (3, –1), 84]**
(A) parallelogram, which is neither a rhombus nor a rectangle
(B) square
(C) rectangle, but not a square
(D) rhombus, but not a square

2. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $(2\vec{a}+\vec{b}).\left[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})\right]$ is **[IIT-JEE-2010, Paper-1, (3, 0), 84]**

3. Two adjacent sides of a parallelogram ABCD are given by **[IIT-JEE-2010, Paper-2, (5, –2), 79]**
$\overline{AB} = 2\hat{i}+10\hat{j}+11\hat{k}$ and $\overline{AD} = -\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD′ . If AD′ makes a right angle with the side AB, then the cosine of the angle α is given by
(A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

4. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is **[IIT-JEE-2010, Paper-1, (3, –1), 84]**
(A) x + 2y – 2z = 0 (B) 3x + 2y – 2z = 0 (C) x – 2y + z = 0 (D) 5x + 2y – 4z = 0

5. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is **[IIT-JEE-2010, Paper-1, (3, –1), 84]**
(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

6. If the distance between the plane Ax – 2y + z = d and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then |d| is
[IIT-JEE-2010, Paper-1, (3, 0), 84]

7. If the distance of the point P(1, –2, 1) from the plane x + 2y – 2z = α, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is **[IIT-JEE-2010, Paper-2, (5, –2), 79]**
(A) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (B) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$ (C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (D) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

8. Match the statements in **Column-I** with those in **Column-II.** **[IIT-JEE-2010, Paper-2, (8, 0), 79]**

Column-I		**Column-II**
(A)	A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	(p) -4
(B)	The values of x satisfying $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}\left(\frac{3}{5}\right)$ are	(q) 0
(C)	Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b}=0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c})=0$ and $2|\vec{b}+\vec{c}|=|\vec{b}-\vec{a}|$. If $\vec{a}=\mu\vec{b}+4\vec{c}$ then possible value of μ are	(r) 4
(D)	Let f be the function on $[-\pi, \pi]$ given by	(s) 5
	$f(0)=9$ and $f(x)=\frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ for $x \neq 0$. The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$ is	(t) 6

9. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\hat{i}-3\hat{j}+3\hat{k}$ (B) $-3\hat{i}-3\hat{j}-\hat{k}$ (C) $3\hat{i}-\hat{j}+3\hat{k}$ (D) $\hat{i}+3\hat{j}-3\hat{k}$

10*. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

(A) $\hat{j}-\hat{k}$ (B) $-\hat{i}+\hat{j}$ (C) $\hat{i}-\hat{j}$ (D) $-\hat{j}+\hat{k}$

11. Let $\vec{a}=-\hat{i}-\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}\ .\ \vec{a}=0$, then the value of $\vec{r}\ .\ \vec{b}$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

12. Match the statements given in **Column-I** with the values given in **Column-II** **[IIT-JEE 2011, Paper-2, (8, 0), 80]**

Column-I		**Column-II**
(A)	If $\vec{a}=\hat{j}+\sqrt{3}\ \hat{k}$, $\vec{b}=-\hat{j}+\sqrt{3}\ \hat{k}$ and $\vec{c}=2\sqrt{3}\ \hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B)	If $\int_a^b (f(x)-3x)\ dx = a^2-b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C)	The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec\ (\pi x)\ dx$ is	(r) $\frac{\pi}{3}$
(D)	The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $|z| = 1$, $z \neq 1$ is given by	(s) π
		(t) $\frac{\pi}{2}$

13. The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1, –1, 4) with the plane 5x – 4y – z = 1. If S is the foot of the perpendicular drawn from the point T(2, 1,4) to QR, then the length of the line segment PS is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

14. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a}-\vec{b}|^2 + |\vec{b}-\vec{c}|^2 + |\vec{c}-\vec{a}|^2 = 9$, then $|2\vec{a} + 5\vec{b} + 5\vec{c}|$ is

[IIT-JEE 2012, Paper-1, (4, 0), 70]

15. The equation of a plane passing through the line of intersection of the planes x + 2y + 3z = 2 and x – y + z = 3 and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 5x – 11y + z = 17 (B) $\sqrt{2}x + y = 3\sqrt{2} - 1$ (C) $x + y + z = \sqrt{3}$ (D) $x - \sqrt{2}y = 1 - \sqrt{2}$

16. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}| = \sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b}) \cdot (-7\hat{i}+2\hat{j}+3\hat{k})$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 0 (B) 3 (C) 4 (D) 8

17*. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are) **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(A) y + 2z = –1 (B) y + z = –1 (C) y – z = –1 (D) y – 2z = –1

18. Let $\overrightarrow{PR} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ} = \hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT} = \hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) 5 (B) 20 (C) 10 (D) 30

19. Perpendicular are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane x + y + z = 3. The feet of perpendiculars lie on the line **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$ (C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

20.* A line l passing through the origin is perpendicular to the lines

l_1 : (3 + t) $\hat{i}$ + (– 1 + 2t) $\hat{j}$ + (4 + 2t) $\hat{k}$, $-\infty < t < \infty$

l_2 : (3 + 2s) $\hat{i}$ + (3 + 2s) $\hat{j}$ + (2 + s) $\hat{k}$, $-\infty < s < \infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are) **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ (B) (–1, ,–1, 0) (C) (1, 1, 1) (D) $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$

21. Consider the set of eight vectors V = $\{a\hat{i}+b\hat{j}+c\hat{k} : a,b,c \in \{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

22.* Two lines L_1 : x = 5, $\frac{y}{3-\alpha} = \frac{z}{-2}$ and L_2 : x = α, $\frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 1 (B) 2 (C) 3 (D) 4

23. Match List I with List II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

List - I		**List - II**	
(P)	Volume of parallelopiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}),3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	**(1)**	100
(Q)	Volume of parallelepiped determined by vectors $\vec{a},\vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}),(\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	**(2)**	30
(R)	Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $\left(2\vec{a}+3\vec{b}\right)$ and $(\vec{a}-\vec{b})$ is	**(3)**	24
(S)	Area of a paralelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	**(4)**	60

Codes :

	P	Q	R	S
(A)	4	2	3	1
(B)	2	3	1	4
(C)	3	4	1	2
(D)	1	4	3	2

24. Consider the lines $L_1 : \frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{1}$, $L_2\ \frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}$: and the planes P_1 : 7x + y + 2z = 3, P_2 : 3x + 5y – 6z = 4. Let ax + by + cz = d the equation of the plane passing through the point of intersection of lines L_1 and L_2, and perpendicular to planes P_1 and P_2.

Match List - I with List- II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

List- I		**List- II**	
P.	a =	1.	13
Q.	b =	2.	–3
R.	c =	3.	1
S.	d =	4.	–2

Codes :

	P	Q	R	S
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3

25*. Let $\vec{x},\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$.

If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is a nonzero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(A) $\vec{b}=(\vec{b}.\vec{z})(\vec{z}-\vec{x})$ (B) $\vec{a}=(\vec{a}.\vec{y})(\vec{y}-\vec{z})$ (C) $\vec{a}.\vec{b}=-(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a}=(\vec{a}.\vec{y})(\vec{z}-\vec{y})$

26. From a point P(λ,λ,λ), perpendiculars PQ and PR are drawn respectively on the lines y = x, z = 1 and y = –x, z = –1. If P is such that ∠QPR is a right angle, then the possible value(s) of λ is(are)

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) $\sqrt{2}$ (B) 1 (C) –1 (D) $-\sqrt{2}$

27. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

28. **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

List I		**List II**	
P.	Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1]$, $x \neq \pm \frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals	**1.**	1
Q.	Let $A_1, A_2, \ldots\ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, k = 1, 2,...., n. If $\left\lvert\sum_{k=1}^{n-1}(\overrightarrow{a_k}\times\overrightarrow{a_{k+1}})\right\rvert=\left\lvert\sum_{k=1}^{n-1}(\overrightarrow{a_k}.\overrightarrow{a_{k+1}})\right\rvert$, then the minimum value of n is	**2.**	2
R.	If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line x + y = 8, then the value of h is	**3.**	8
S.	Number of positive solutions satisfying the equation is $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$	**4.**	9

	P	**Q**	**R**	**S**
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

29*✍. In R^3, consider the planes P_1 : y = 0 and P_2 : x + z = 1. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relation is (are) true ?

[JEE(Advanced)2015,P-1 (4, –2)/ 88]

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$
(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

30*✍. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes P_1 : x + 2y – z + 1 = 0 and P_2 : 2x – y + z – 1 = 0. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M? **[JEE(Advanced)2015,P-1 (4, –2)/ 88]**

(A) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$ (C) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (D) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

31*. Let ΔPQR be a triangle. Let $\vec{a}=\overrightarrow{QR}, \vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}|=4\sqrt{3}$ and $\vec{b}.\vec{c}=24$, then which of the following is(are) true? **[JEE(Advanced)2015, P-1 (4, –2)/88]**

(A) $\frac{|\vec{c}|^2}{2}-|\vec{a}|=12$ (B) $\frac{|\vec{c}|^2}{2}+|\vec{a}|=30$ (C) $|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=48\sqrt{3}$ (D) $\vec{a}.\vec{b}=-72$

32. **Column- I** **Column-II**

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

	Column- I		Column-II
(A)	In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\ \beta$, then possible value(s) of $\|\alpha\|$ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\ge 1\end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value(s) of n is (are)	(R)	3
(D)	Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5,q, b is an arithmetic progression, then the value(s) of $\|q-a\|$ is (are)	(S)	4
		(T)	5

33. **Column- I** **Column-II**

	Column- I		Column-II
(A)	In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2-b^2)=c^2$ and $\lambda = \dfrac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda)=0$ is (are)	(P)	1
(B)	In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1+\cos 2X-2\cos 2Y=2\sin X\sin Y$, then possible value(s) of $\dfrac{a}{b}$ is (are)	(Q)	2
(C)	In R^2, let $\sqrt{3}\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$ and $\beta\ \hat{i}+(1-\beta)\ \hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is, $\dfrac{3}{\sqrt{2}}$ then possible value(s) of $\|\beta\|$ is (are)	(R)	3
(D)	Suppose that $F(\alpha)$ denotes the area of the region bounded by $x=0$, $x=2$, $y^2=4x$ and $y=\|\alpha x-1\|+\|\alpha x-2\|+\alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha)+\frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	(S)	5
		(T)	6

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

34. Suppose that $\vec{p},\vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p},\vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}),(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z, respectively, then the value of $2x+y+z$ is

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

35*. Consider a pyramid OPQRS located in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid point T of diagonal OQ such that TS = 3. Then

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$ **[JEE (Advanced) 2016, Paper-1, (4, –2)/62]**

(B) the equation of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

36. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

[JEE (Advanced) 2016, Paper-2, (3, –1)/62]

(A) $x + y - 3z = 0$ (B) $3x + z = 0$ (C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

37*. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in R^3 and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector $\vec{u}$ in R^3 such that $|\hat{u} \times \vec{u}| = 1$ and $\hat{w}.(\hat{u} \times \vec{u}) = 1$. Which of the following statements(s) is (are) correct?

(A) There is exactly one choice for $\vec{u}$

(B) There are infinitely many choices for such $\vec{u}$ **[JEE (Advanced) 2016, Paper-2, (4, –2)/62]**

(C) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$

(D) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

38. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

Then the triangle PQR has S as its **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

(A) centroid (B) orthocenter (C) incentre (D) circumcenter

39. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

(A) $14x + 2y - 15z = 1$ (B) $-14x + 2y + 15z = 3$

(C) $14x - 2y + 15z = 27$ (D) $14x + 2y + 15z = 31$

Comprehension (Q.40 & 41)

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

40. If the triangle PQR varies, then the minimum value of $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ is

[JEE(Advanced) 2017, Paper-2,(3, 0)/61]

(A) $-\frac{3}{2}$ (B) $\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

41. $|\overrightarrow{OX} \times \overrightarrow{OY}| =$ **[JEE(Advanced) 2017, Paper-2,(3, 0)/61]**

(A) sin(P + Q) (B) sin(P + R) (C) sin(Q + R) (D) sin2R

42*. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018, Paper-1,(4, –2),60]**

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, – 1

(B) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60º

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$

43. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b} = 0$. For some x, y ∈ R, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of 8 $\cos^2\alpha$ is _______. **[JEE(Advanced) 2018, Paper-1, (3, 0), 60]**

44. Let P be a point in the first octant, whose image Q in the plane x + y = 3 (that is, the line segment PQ is perpendicular to the plane x + y = 3 and the mid-point of PQ lies in the plane x + y = 3) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _______ . **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

45. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is ______ . **[JEE(Advanced) 2018, Paper-2,(3, 0), 60]**

46*. Let L_1 and L_2 denote the lines $\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k})$, λ ∈ R and $\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k})$, μ ∈ R respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ? **[JEE(Advanced) 2019, Paper-1,(4, –1)/62]**

(A) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R

(B) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R

(C) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R

(D) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R

47. Three lines are given by

$\vec{r} = \lambda\hat{i},\ \lambda \in R$

$\vec{r} = \mu(\hat{i} + \hat{j})$, n ∈ R and $\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k}),\ \nu \in R$

Let the lines cut the plane x + y + z = 1 at he points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\underline{\Delta})^2$ equals ……… **[JEE(Advanced) 2019, Paper-1,(4, –1)/62]**

48*. Three lines

$L_1 : \vec{r} = \lambda \hat{i}$, $\lambda \in R$

$L_2 : \vec{r} = \hat{k} + \mu \hat{j}$, $\mu \in R$ and

$L_3 : \vec{r} = \hat{i} + \hat{j} + \nu \hat{k}$, $\nu \in R$.

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P,Q and R are collinear. **[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]**

(A) $\hat{k} + \frac{1}{2}\hat{j}$ (B) $\hat{k} + \hat{j}$ (C) $\hat{k}$ (D) $\hat{k} - \frac{1}{2}\hat{j}$

49. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, α, $\beta \in R$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $(\vec{c} - (\vec{a} \times \vec{b})).\vec{c}$ equal to

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let $\vec{a} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$ and $\vec{a}.\vec{b} = 3$ is

[AIEEE 2010 (4, –1), 144]

(1) $2\hat{i} - \hat{j} + 2\hat{k}$ (2) $\hat{i} - \hat{j} - 2\hat{k}$ (3) $\hat{i} + \hat{j} - 2\hat{k}$ (4) $-\hat{i} + \hat{j} - 2\hat{k}$

2. If the vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then (λ , μ) = **[AIEEE 2010 (4, –1), 144]**

(1) (2, – 3) (2) (–2, 3) (3) (3, – 2) (4) (–3, 2)

3. **Statement -1 :** The point A(3, 1, 6) is the mirror image of the point B(1, 3, 4) in the plane x – y + z = 5.

Statement -2 : The plane x – y + z = 5 bisects the line segment joining A(3, 1,6) and B(1, 3, 4).

(1) Statement -1 is true, Statement-2 is true; Statement -2 is not a correct explanation for Statement -1.

(2) Statement-1 is true, Statement-2 is false. **[AIEEE 2009 (4, –1), 144]**

(3) Statement -1 is false, Statement -2 is true.

(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

4. A line AB in three-dimensional space makes angles 45º and 120º with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equal

[AIEEE 2010 (4, –1), 144]

(1) 45º (2) 60º (3) 75º (4) 30º

5. If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$, then the value of $(2\vec{a} - \vec{b}) . [(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})]$ is:

[AIEEE 2011, I, (4, –1), 120]

(1) – 5 (2) –3 (3) 5 (4) 3

6. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying : $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to : **[AIEEE 2011, I, (4, –1), 120]**

(1) $\vec{b} - \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{d}}\right)\vec{c}$ (2) $\vec{c} + \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (3) $\vec{b} + \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (4) $\vec{c} - \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$

7. If the vector $p\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + q\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + r\hat{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of pqr – (p+q+r) is-

[AIEEE 2011, II, (4, –1), 120]

(1) 2 (2) 0 (3) –1 (4) –2

8✍. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is : **[AIEEE 2011, II, (4, –1), 120]**
(1) $\vec{a}$ (2) $\vec{c}$ (3) $\vec{0}$ (4) $\vec{a} + \vec{c}$

9. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane x + 2y + 3z = 4 is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals :
[AIEEE 2011, I, (4, –1), 120]
(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) $\frac{2}{5}$ (4) $\frac{5}{3}$

10. **Statement-1 :** The point A(1, 0, 7) is the mirror image of the point B(1, 6, 3) in the line : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ **[AIEEE 2011, I, (4, –1), 120]**
Statement-2 : The line : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining A(1, 0, 7) and B(1, 6, 3).
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(3) Statement-1 is true, Statement-2 is false.
(4) Statement-1 is false, Statement-2 is true.

11. The distance of the point (1, –5, 9) from the plane x – y + z = 5 measured along a straight line x = y = z is : **[AIEEE 2011, II, (4, –1), 120]**
(1) $10\sqrt{3}$ (2) $5\sqrt{3}$ (3) $3\sqrt{10}$ (4) $3\sqrt{5}$

12. The length of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is :
[AIEEE 2011, II, (4, –1), 120]
(1) $\sqrt{29}$ (2) $\sqrt{33}$ (3) $\sqrt{53}$ (4) $\sqrt{66}$

13. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is : **[AIEEE-2012, (4, –1)/120]**
(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

14. A equation of a plane parallel to the plane x – 2y + 2z – 5 = 0 and at a unit distance from the origin is :
[AIEEE 2012, (4, –1), 120]
(1) x – 2y + 2z – 3 = 0 (2) x – 2y + 2z + 1 = 0 (3) x – 2y + 2z – 1 = 0 (4) x – 2y + 2z + 5 = 0

15. If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to :
[AIEEE 2012, (4, –1), 120]
(1) – 1 (2) $\frac{2}{9}$ (3) $\frac{9}{2}$ (4) 0

16. Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{q}$, $\overrightarrow{AD} = \vec{p}$ and $\angle BAD$ be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then is given by :
[AIEEE-2012, (4, –1)/120]
(1) $\vec{r} = 3\vec{q} - \frac{3(\vec{p}\,.\,\vec{q})}{(\vec{p}\,.\,\vec{p})}\,\vec{p}$ (2) $\vec{r} = -\vec{q} + \left(\frac{\vec{p}\,.\,\vec{q}}{\vec{p}\,.\,\vec{p}}\right)\vec{p}$ (3) $\vec{r} = \vec{q} - \left(\frac{\vec{p}\,.\,\vec{q}}{\vec{p}\,.\,\vec{p}}\right)\vec{p}$ (4) $\vec{r} = -3\vec{q} + \frac{3(\vec{p}\,.\,\vec{q})}{(\vec{p}\,.\,\vec{p})}\,\vec{p}$

17. Distance between two parallel planes 2x + y + 2z = 8 and 4x + 2y + 4z + 5 = 0 is **[AIEEE - 2013, (4, –1),360]**

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3) $\frac{7}{2}$ (4) $\frac{9}{2}$

18. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k can have **[AIEEE - 2013, (4, –1),360]**

(1) any value (2) exactly one value (3) exactly two values (4) exactly three values

19. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is **[AIEEE - 2013, (4, –1),360]**

(1) $\sqrt{18}$ (2) $\sqrt{72}$ (3) $\sqrt{33}$ (4) $\sqrt{45}$

20. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane 2x – y + z + 3 = 0 is the line : **[JEE(Main)2014,(4, – 1),120]**

(1) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$ (2) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

(3) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ (4) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

21✍. The angle between the lines whose direction cosines satisfy the equations $\ell + m + n = 0$ and $\ell^2 = m^2 + n^2$ is **[JEE(Main)2014,(4, – 1), 120]**

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

22. If $\left[\vec{a} \times \vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}\right] = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]^2$ then λ is equal to **[JEE(Main)2014,(4, – 1), 120]**

(1) 0 (2) 1 (3) 2 (4) 3

23. The distance of the point (1,0,2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane x – y + z = 16, is **[JEE(Main)2015,(4, – 1), 120]**

(1) $2\sqrt{14}$ (2) 8 (3) $3\sqrt{21}$ (4) 13

24. The equation of the plane containing the line 2x – 5y + z = 3, x + y + 4z = 5 and parallel to the plane x + 3y + 6z = 1, is **[JEE(Main)2015,(4, – 1), 120]**

(1) 2x + 6y + 12z = 13 (2) x + 3y + 6z = –7

(3) x + 3y + 6z = 7 (4) 2x + 6y + 12z = –13

25. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\,\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is **[JEE(Main)2015,(4,– 1), 120]**

(1) $\frac{2\sqrt{2}}{3}$ (2) $\frac{-\sqrt{2}}{3}$ (3) $\frac{2}{3}$ (4) $\frac{-2\sqrt{3}}{3}$

26. If the line, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, lx + my – z = 9, then $l^2 + m^2$ is equal to **[JEE(Main)2016,(4, – 1),120]**

(1) 18 (2) 5 (3) 2 (4) 26

27. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is **[JEE(Main)2016,(4, – 1), 120]**

(1) $\frac{\pi}{2}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{6}$ (4) $\frac{3\pi}{4}$

28. The distance of the point (1, –5, 9) from the plane x – y + z = 5 measured along the line x = y = z is **[JEE(Main)2016,(4, – 1), 120]**

(1) $10\sqrt{3}$ (2) $\frac{10}{\sqrt{3}}$ (3) $\frac{20}{3}$ (4) $3\sqrt{10}$

29. If the image of the point P(1, –2, 3) in the plane, 2x + 3y – 4z + 22 = 0 measured parallel to the line, $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to : **[JEE(Main)2017,(4, – 1), 120]**

(1) $3\sqrt{5}$ (2) $2\sqrt{42}$ (3) $\sqrt{42}$ (4) $6\sqrt{5}$

30. The distance of the point (1, 3, – 7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{x+2}{-2} = \frac{x-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is **[JEE(Main)2017,(4,– 1), 120]**

(1) $\frac{20}{\sqrt{74}}$ (2) $\frac{10}{\sqrt{83}}$ (3) $\frac{5}{\sqrt{83}}$ (4) $\frac{10}{\sqrt{74}}$

31. Let $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c}-\vec{a}| = 3$, $|(\vec{a}\times\vec{b})\times\vec{c}| = 3$ and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30º. Then $\vec{a}.\vec{c}$ is equal to **[JEE(Main)2017,(4, – 1), 120]**

(1) $\frac{25}{8}$ (2) 2 (3) 5 (4) $\frac{1}{8}$

32. If L_1 is the line of intersection of the planes 2x – 2y + 3z – 2 = 0, x – y + z + 1 = 0 and L_2 is the line of intersection of the planes x + 2y – z – 3 = 0, 3x – y + 2z – 1 = 0, then the distance of the origin from the plane, containing the lines L_1 and L_2, is : **[JEE(Main)2018, (4, –1), 120]**

(1) $\frac{1}{2\sqrt{2}}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{1}{4\sqrt{2}}$ (4) $\frac{1}{3\sqrt{2}}$

33. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{j}+\hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b} = 24$, then $|\vec{u}|^2$ is equal to : **[JEE(Main)2018, (4, –1), 120]**

(1) 256 (2) 84 (3) 336 (4) 315

34. The length of the projection of the line segment joining the points (5, – 1, 4) and (4,–1,3) on the plane, x + y + z = 7 is : **[JEE(Main)2018, (4, –1), 120]**

(1) $\frac{1}{3}$ (2) $\sqrt{\frac{2}{3}}$ (3) $\frac{2}{\sqrt{3}}$ (4) $\frac{2}{3}$

35. If the lines x = ay + b, z = cy + d and x = a'z + b', y = c'z + d' are perpendicular, then : **[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]**

(1) ab' + bc' + 1 = 0 (2) bb' + cc' + 1 = 0 (3) cc' + a + a' = 0 (4) aa' + c + c' = 0

36. Let $\vec{a} = \hat{i}+\hat{j}+\sqrt{2}\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+\sqrt{2}\hat{k}$ and $\vec{c} = 5\hat{i}+\hat{j}+\sqrt{2}\hat{k}$ be three vectors such that the projection vector of $\vec{b}$ on $\vec{a}$ is $\vec{a}$. If $\vec{a}+\vec{b}$ is perpendicular to $\vec{c}$, then $|\vec{b}|$ is equal to : **[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]**

(1) $\sqrt{22}$ (2) 4 (3) $\sqrt{32}$ (4) 6

37. A tetrahedron has vertices P(1,2,1), Q(2,1,3), R(–1,1,2) and O(0,0,0) the angle between the faces OPQ and PQR is : **[JEE(Main) 2019, Online (12-01-19),P-1 (4, – 1), 120]**

(1) $\cos^{-1}\left(\frac{19}{35}\right)$ (2) $\cos^{-1}\left(\frac{7}{31}\right)$ (3) $\cos^{-1}\left(\frac{17}{31}\right)$ (4) $\cos^{-1}\left(\frac{9}{35}\right)$

38. Let S be the set of all real values of λ such that a plane passing through the points $(-\lambda^2, 1, 1)$, $(1, -\lambda^2, 1)$ and $(1, 1, -\lambda^2)$ also passes through the point (–1, –1, 1). Then S is equal to -
[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]

(1) {1, –1} (2) $\{\sqrt{3}\}$ (3) $\{\sqrt{3}, -\sqrt{3}\}$ (4) {3, –3}

39. The magnitude of the projection of the vector $2\hat{i}+3\hat{j}+\hat{k}$ on the vector perpendicular to the plane containing the vectors $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$, is :

(1) $3\sqrt{6}$ (2) $\sqrt{6}$ (3) $\sqrt{\frac{3}{2}}$ (4) $\frac{\sqrt{3}}{2}$

[JEE(Main) 2019, Online (08-04-19),P-1 (4, – 1), 120]

40. If the volume of parallelepiped formed by the vectors $\hat{i}+\lambda\hat{j}+\hat{k}, \hat{j}+\lambda\hat{k}$ and $\lambda\hat{i}+\hat{k}$ is minimum , then λ is equal to : **[JEE(Main) 2019, Online (12-04-19),P-1 (4, – 1), 120]**

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) $\frac{1}{\sqrt{3}}$ (4) $-\frac{1}{\sqrt{3}}$

41. The vertices B and C of a ΔABC lie on the line, $\frac{x+2}{3}-\frac{y-1}{0}=\frac{2}{4}$ such that BC = 5 units. Then the area (in sq. units) of this triangle, given that the point A(1, –1, 2), is :
[JEE(Main) 2019, Online (09-04-19),P-2 (4, – 1), 120]

(1) $5\sqrt{17}$ (2) 6 (3) $\sqrt{34}$ (4) $2\sqrt{34}$

42. Let A(3, 0, –1), B(2, 10, 6) and C(1,2,1) be the vertices of a triangle and M be the midpoint of AC. If G divides BM in the ratio, 2 : 1, then cos($\angle$GOA) (O being the origin) is equal to :
[JEE(Main) 2019, Online (10-04-19),P-1 (4, – 1), 120]

(1) $\frac{1}{6\sqrt{10}}$ (2) $\frac{1}{2\sqrt{15}}$ (3) $\frac{1}{\sqrt{15}}$ (4) $\frac{1}{\sqrt{30}}$

43. A perpendicular is drawn from a point on the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{1}$ to the plane x + y + z = 3 such that the foot of the perpendicular Q also lies on the plane x – y + z = 3. Then the co-ordinates of Q are :

(1) (4, 0, –1) (2) (2, 0, 1) (3) (–1, 0, 4) (4) (1, 0, 2)

[JEE(Main) 2019, Online (10-04-19),P-2 (4, – 1), 120]

44. A vector $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ $(\alpha, \beta, \in R)$ lies in the plane of the vectors, $\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$.
If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then: **[JEE(Main) 2020, Online (07-01-20),P-1 (4, – 1), 120]**

(1) $\vec{a}.\hat{k}+4=0$ (2) $\vec{a}.\hat{k}+2=0$ (3) $\vec{a}.\hat{i}+1=0$ (4) $\vec{a}.\hat{i}+3=0$

45. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectros such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. If $\lambda=\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ and $\vec{d}=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$, then the ordered pair, $(\lambda, \vec{d})$ is equal to:

[JEE(Main) 2020, Online (07-01-20),P-2 (4, –1), 120]

(1) $\left(\frac{3}{2}, 3\vec{a}\times\vec{c}\right)$ (2) $\left(-\frac{3}{2}, 3\vec{c}\times\vec{b}\right)$ (3) $\left(\frac{3}{2}, 3\vec{b}\times\vec{c}\right)$ (4) $\left(-\frac{3}{2}, 3\vec{a}\times\vec{b}\right)$

46. If the distance between the plane, 23x –10y –2z +48 = 0 and the plane containing the lines $\frac{x+1}{2}=\frac{y-3}{4}=\frac{z+1}{3}$ and $\frac{x+3}{2}=\frac{y+2}{6}=\frac{z-1}{\lambda}$ $(\lambda \in R)$ is equal to $\frac{k}{\sqrt{633}}$, then k is equal to ___________

[JEE(Main) 2020, Online (09-01-20),P-2 (4, 0), 120]

Answers

EXERCISE - 1

PART – I

Section (A) :

A-1. (i) $\dfrac{2\vec{c}+\vec{b}-3\vec{a}}{3}$ (ii) $\dfrac{3\vec{c}+5\vec{a}-8\vec{d}}{5}$ (iii) $\dfrac{5\vec{b}+\vec{a}-6\vec{c}}{6}$

A-2. $5\hat{i}+\hat{j}+4\hat{k}$, $4\hat{i}-\hat{j}+5\hat{k}$ **A-4.** $\left(\dfrac{6}{7},\dfrac{2}{7},\dfrac{3}{7}\right)$, $\left(\dfrac{-6}{7},\dfrac{-2}{7},\dfrac{-3}{7}\right)$

A-5. (i) 40, 38, 32 (ii) $\sqrt{41}$, $\sqrt{26}$, $\sqrt{17}$ **A-6.** $-\dfrac{1}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}$ or $\dfrac{2}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}}$

Section (B) :

B-1. $\angle A = \cos^{-1}\sqrt{\dfrac{35}{41}}$, $\angle B = \cos^{-1}\sqrt{\dfrac{6}{41}}$, $\angle C = \dfrac{\pi}{2}$

B-3. **(i)** $\dfrac{10}{\sqrt{6}}$ (ii) $\dfrac{7}{\sqrt{3}}$ (iii) $2\sqrt{2}-2$ **B-6.** 0

B-8. (a) $-\hat{i}+\hat{j}+\hat{k}$ (b) $\dfrac{6}{\sqrt{19}}(6\hat{i}-\hat{j}+\hat{k})$ (c) $\dfrac{2\pi}{3}$ **B-9.** – 169

B-10. $\theta = \cos^{-1}\dfrac{2}{3\sqrt{713}}$ **B-12.** (i) $\cos^{-1}\left(\dfrac{1}{6}\right)$ (ii) 60º **B-13.** $\pm(4\hat{i}-6\hat{j}+12\hat{k})$

B-14. $3\left(-\hat{i}+\hat{j}+\hat{k}\right)$ **B-15.** (b) 5 unit sq.

Section (C) :

C-1. $\left(\dfrac{1}{2},\dfrac{5}{2},0\right)$ **C-2.** (1, 3, 5) **C-3.** (i) $\dfrac{x-2}{1}=\dfrac{y+1}{1}=\dfrac{z-4}{-2}$ (ii) $\vec{r}=(2\hat{i}-2\hat{j}+\hat{k})+\lambda(3\hat{i}+4\hat{j}-\hat{k})$

C-4. $\sqrt{26}$ **C-5.** 3 **C-6.** $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{-1}$, $\dfrac{x}{-1}=\dfrac{y}{1}=\dfrac{z}{-2}$

C-7. $r=\dfrac{a+b+c}{3}$ **C-8.** $\dfrac{6}{\sqrt{5}}$ unit **C-9.** A = (3, 8, 3), B = (–3, – 7, 6), AB = $3\sqrt{30}$

C-10. $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+t\ (\hat{j}-\hat{k})$, where t is parameter

Section (D) :

D-2. $\sin\alpha\cos\alpha$ **D-3.** $\dfrac{1}{4}(a_1^2+a_2^2+a_3^2)\ (b_1^2+b_2^2+b_3^2)$

D-4. (a) Coplanar (b) Non-coplanar **D-5.** (i) 1/2 unit^3 (ii) $\dfrac{3}{\sqrt{35}}$ unit

D-6. (i) No (ii) Yes **D-7.** x = 1 **D-8.** $\vec{v} = 0$

D-9. (i) $p = 0; q = 10; r = -3$ (ii) -100 **D-10.** $\vec{x} = \vec{q} - \dfrac{(\vec{p}\ .\ \vec{q})\ \vec{p}}{2|\vec{p}|^2}$

D-11. $\vec{x} = \dfrac{\vec{d}\times(\vec{c}\times\vec{d}) - 2|\vec{d}|^2\,\vec{a}}{\vec{d}.\vec{a}}$

Section (E) :

E-1. (i) $4x - y - z + 1 = 0$ (ii) $x + 2y + 3z - 2 = 0$ (iii) $x + y + z - 4 = 0$ (iv) $x + y + z - 6 = 0$

E-2. 3 : 2, (0,13/5, 2) **E-3.** $x^2 + y^2 + z^2 = 9$ **E-4.** $\vec{r}\ .(4\hat{i} - 2\hat{j} - 5\hat{k}) = 45$

E-5. $\sin^{-1}\dfrac{4}{\sqrt{30}}$ **E-6.** $\dfrac{x-3}{3} = \dfrac{y+2}{4} = \dfrac{z-5}{5}$ **E-7.** $\dfrac{x-4}{9} = \dfrac{y+1}{-1} = \dfrac{z-7}{-3}$

E-8. $\pi/2$ **E-9.** $11x - y - 3z = 35$ **E-10.** 7 **E-11.** 0

E-12. $8x - 13y + 15z + 13 = 0$

E-13. $\vec{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + t(6\hat{i} - 3\hat{j} + 5\hat{k})$, $\left(\dfrac{76}{35}, \dfrac{-108}{35}, \dfrac{-170}{35}\right)$ **E-14.** $x - y + 3z - 2 = 0$; (3, 1, 0) ; $\sqrt{11}$

E-15. $x + y \pm \sqrt{2}\ z = 1$ **E-16.** $\dfrac{5}{3}$ unit **E-17.** 1 **E-18.** (i) $\vec{r}\ .\hat{n} = \pm p$ (ii) $\vec{r}\ .(\vec{a}\,q - p\vec{b}) = 0$

PART – II

Section (A) :

A-1. (C) **A-2.** (C) **A-3.** (B) **A-4.** (B)

Section (B) :

B-1. (D) **B-2.** (B) **B-3.** (C) **B-4.** (C) **B-5.** (C) **B-6.** (B) **B-7.** (B)
B-8. (C)

Section (C) :

C-1. (A) **C-2.** (C) **C-3.** (A) **C-4.** (A) **C-5.** (D) **C-6.** (C)

Section (D) :

D-1. (C) **D-2.** (C) **D-3.** (C) **D-4.** (B) **D-5.** (A) **D-6.** (D)
D-7. (D) **D-8.** (D) **D-9.** (B) **D-10.** (B) **D-11.** (B) **D-12.** (B)

Section (E) :

E-1. (A) **E-2.** (B) **E-3.** (D) **E-4.** (A) **E-5.** (D) **E-6.** (A)
E-7. (D) **E-8.** (A) **E-9.** (A) **E-10.** (C)

PART - III

1. (A) → (r), (B) → (q), (C) → (q), (D) → (s)
2. (A) → (q), (B) → (p), (C) → (s), (D) → (r)
3. (A) → (q), (B) → (p, r), (C) → (r), (D) → (s)

EXERCISE - 2

PART – I

1. (D) **2.** (C) **3.** (C) **4.** (A) **5.** (C) **6.** (A) **7.** (C)
8. (C) **9.** (D) **10.** (B) **11.** (C) **12.** (A) **13.** (C) **14.** (A)
15. (A) **16.** (C) **17.** (A) **18.** (D) **19.** (A) **20.** (B) **21.** (A)
22. (A) **23.** (B) **24.** (A)

PART – II

1. 33.54 **2.** 19.00 **3.** 06.00 **4.** 69.00 **5.** 04.50 **6.** 02.00
7. 72.00 **8.** 34.00 **9.** 00.66 or 00.67 **10.** 04.24 **11.** 36.00 **12.** 01.75
13. 10.66 or 10.67 **14.** 32.00 **15.** 22.62 or 22.63 **16.** 00.41 or 00.42
17. 13.00 **18.** 07.20 **19.** 04.00

PART – III

1. (AB) **2.** (ABC) **3.** (ABCD) **4.** (BCD) **5.** (BD) **6.** (ABC)
7. (ABCD) **8.** (AC) **9.** (BD) **10.** (BC) **11.** (BC)
12. (B) **13.** (AB) **14.** (ABCD) **15.** (ABCD) **16.** (BD) **17.** (D)
18. (AB) **19.** (AC) **20.** (AD) **21.** (AC) **22.** (AD) **23.** (AB)
24. (ABC) **25.** (ABC) **26.** (AD) **27.** (BCD) **28.** (AB) **29.** (ABCD)
30. (AD) **31.** (ABD) **32.** (ACD)

PART – IV

1. (A) **2.** (A) **3.** (D) **4.** (B)
5. (C) **6.** (A) **7.** (A) **8.** (B) **9.** (A) **10.** (A) **11.** (C)
12. (C) **13.** (A) **14.** (A) **15.** (A)

EXERCISE - 3

PART – I

1. (A) **2.** 5 **3.** (B) **4.** (C) **5.** (A)
6. 6 **7.** (A) **8.** (A) $\rightarrow$ (t), (B) $\rightarrow$ (p, r), (C) $\rightarrow$ (q) (JEE given q, s) (D) $\rightarrow$ (r)
9. (C) **10*.** (A, D) **11.** 9 **12.** (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s), (D) $\rightarrow$ (t)
13. (A) **14.** 3 **15.** (A) **16.** (C) **17*.** (BC) **18.** (C) **19.** (D)
20.* (BD) **21.** ${}^8C_3 - 24 = 32$ **22.*** (AD) **23.** (C) **24.** (A) **25*.** (ABC)
26. (C) **27.** (4) **28.** (A) **29*.** (B,D) **30*.** (A,B) **31*.** (ACD)
32. (A) $\rightarrow$ P,Q ; (B) $\rightarrow$ P, Q ; (C) $\rightarrow$ P,Q,S,T ; (D) $\rightarrow$ Q, T
33. (A)$\rightarrow$P,R,S ; (B)$\rightarrow$P ; (C)$\rightarrow$P,Q ; (D)$\rightarrow$S, T **34.** BONUS **35*.** (BCD)
36. (C) **37*.** (BC) **38.** (B) **39.** (D) **40.** (A) **41.** (A) **42*.** (CD)
43. (3) **44.** (8) **45.** (0.5) **46.** (ABC) **47.** (0.75) **48.** (AD) **49.** (18.00)

PART – II

1. (4) **2.** (4) **3.** (1) **4.** (2) **5.** (1) **6.** (4) **7.** (4)
8. (3) **9.** (1) **10.** (2) **11.** (1) **12.** (3) **13.** (3) **14.** (1)
15. (3) **16.** (2) **17.** (3) **18.** (3) **19.** (3) **20.** (3) **21.** (3)
22. (2) **23.** (4) **24.** (3) **25.** (1) **26.** (3) **27.** (3) **28.** (1)
29. (2) **30.** (2) **31.** (2) **32.** (4) **33.** (3) **34.** (2) **35.** (4)
36. (4) **37.** (1) **38.** (3) **39.** (3) **40.** (3) **41.** (3) **42.** (3)
43. (2) **44.** (2) **45.** (4) **46.** 3

High Level Problems (HLP)

1. Using Vectors prove that
(i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

2. Using vectors, prove that the altitudes of a triangle are concurrent.

3. Prove that the direction cosines of a line equally inclined to three mutually perpendicular lines having D.C.'s as ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 ; ℓ_3, m_3, n_3 are $\frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$.

4. If $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$, $D(\vec{d})$ are the position vector of cyclic quadrilateral then find the value of $\frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|}{[(\vec{b}-\vec{a}).(\vec{d}-\vec{a})]} + \frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|}{[(\vec{b}-\vec{c}).(\vec{d}-\vec{c})]}$. (It is given that no angle of cyclic quadrilateral ABCD is right angle)

5. Prove that the volume of tetrahedron bounded by the planes,
$\vec{r}.(m\hat{j}+n\hat{k}) = 0$, $\vec{r}.(n\hat{k}+\ell\hat{i}) = 0$, $\vec{r}.(\ell\hat{i}+m\hat{j}) = 0$, $\vec{r}.(\ell\hat{i}+m\hat{j}+n\hat{k}) = p$ is $\frac{2p^3}{3\ell mn}$.

6. In a Δ ABC, let M be the mid point of segment AB and let D be the foot of the bisector of $\angle$C. Then prove that $\frac{ar(\Delta CDM)}{ar(\Delta ABC)} = \frac{1}{2}\frac{a-b}{a+b} = \frac{1}{2}\tan\frac{A-B}{2}\cot\frac{A+B}{2}$.

7. Let ABC be a triangle.Points M, N and P are taken on the sides AB, BC and CA respectively such that $\frac{AM}{AB} = \frac{BN}{BC} = \frac{CP}{CA} = \lambda$. Prove that the vectors $\overrightarrow{AN}$, $\overrightarrow{BP}$ and $\overrightarrow{CM}$ form a triangle. Also find λ for which the area of the triangle formed by these vectors is the least.

8. In any triangle, show that the perpendicular bisectors of the sides are concurrent.

9. Let ABC be an acute-angled triangle AD be the bisector of $\angle$BAC with D on BC and BE be the altitude from B on AC. Show that $\angle$CED > 45º.

10. In a quadrilateral ABCD, it is given that AB || CD and the diagonals AC and BD are perpendicular to each other. Show that
(a) AD. BC $\geq$ AB.CD (b) AD + BC $\geq$ AB + CD

11. A, B, C, D are four points in space. using vector methods, prove that
$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$ what is the implication of the sign of equality.

12. The direction cosines of a variable line in two near by positions are l, m, n; $l + \delta l$, m + δm, n + δn. Show that the small angle $\delta\theta$ between the two position is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

13. In a ΔABC, prove that distance between centroid and circumcentre is $\sqrt{R^2 - \left(\frac{a^2+b^2+c^2}{9}\right)}$
where R is the circumradius and a, b, c denotes the sides of ΔABC.

14. Prove that the square of the perpendicular distance of a point P (p, q, r) from a line through A(a, b, c) and whose direction cosines are ℓ, m, n is Σ {(q – b) n – (r – c) m}^{2}.

15. **(i)** Let ℓ_1 & ℓ_2 be two skew lines. If P, Q are two distinct points on ℓ_1 and R, S are two distinct points on ℓ_2, then prove that PR can not be parallel to QS.

(ii) A line with direction cosines proportional to (2, 7 – 5) is drawn to intersect the lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$. Find the coordinate of the points of intersection and the length intercepted on it. Also find the equation of intersecting straight line.

16. The base of the pyramid AOBC is an equilateral triangle OBC with each side equal to $4\sqrt{2}$. 'O' is the origin of reference, AO is perpendicular to the plane of Δ OBC and $|\overrightarrow{AO}| = 2$. Then find the cosine of the angle between the skew straight lines one passing through A and the mid point of OB and the other passing through 'O' and the mid point of BC.

17. If D, E, F be three point on BC, CA, AB respectively of a ΔABC. Such that the line AD, BE, CF are concurrent then find the value of $\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}$.

18. Without expanding the determinant, Prove that

$$\begin{vmatrix} na_1+b_1 & na_2+b_2 & na_3+b_3 \\ nb_1+c_1 & nb_2+c_2 & nb_3+c_3 \\ nc_1+a_1 & nc_2+a_2 & nc_3+a_3 \end{vmatrix} = (n^3+1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

19. (i) OABC is a regular tetrahedron D is circumcentre of Δ OAB and E is mid point of edge AC. Prove that DE is equal to half the edge of tetrahedron.

(ii) If V be the volume of a tetrahedron and V′ be the volume of the tetrahedron formed by the centroids and V = k V′ then find the value of k.

20. Given that $\vec{u}=\hat{i}-2\hat{j}+3\hat{k}$ $\vec{v}=2\hat{i}+\hat{j}+4\hat{k}$, $\vec{w}=2\hat{i}+j+3\hat{k}$ and $(\vec{u}.\vec{R}-10)\,i + (\vec{v}.\vec{R}-20)\,j + (\vec{w}.\vec{R}-20)\,\hat{k} = \vec{0}$ then find $\vec{R}$

21. AB , AC and AD are three adjacent edges of a parallelopiped . The diagonal of the parallelopiped passing through A and directed away from it is vector $\vec{a}$. The vector area of the faces containing vertices A , B , C and A , B , D are $\vec{b}$ and $\vec{c}$ respectively i.e. $\overrightarrow{AB}\times\overrightarrow{AC} = \vec{b}$ and $\overrightarrow{AD}\times\overrightarrow{AB}=\vec{c}$. If projection of each edge AB and AC on diagonal vector $\vec{a}$ is $\frac{|\vec{a}|}{3}$, then find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of $\vec{a}$, $\vec{b}$, $\vec{c}$ and $|\vec{a}|$.

22. Prove that if the equation $\vec{r}\times\vec{a}=\vec{b}$ and $\vec{r}\times\vec{c}=\vec{d}$ are consistent $\left(\vec{a}.\vec{d}\neq 0, \vec{b}.\vec{c}\neq 0, \vec{d}\times\vec{b}\neq\vec{0}\right)$ then $\vec{b}.\vec{c}+\vec{a}.\vec{d}=0$

23. If $\vec{a}$ & $\vec{b}$ are two non collinear vector $\vec{a}\cdot\vec{b}\neq 0$ $\underbrace{\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}.............\times(\vec{a}}_{2018\,\text{times}}\times(\vec{a}\times\vec{b})) = \lambda(\vec{a}\times\vec{b})$

24. Let P be an interior point of a triangle ABC and AP, BP, CP meet the sides BC, CA, AB is D, E, F respectively. Show that $\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$.

25. ABCD is a quadrilateral and P, Q are mid-points of CD, AB respectively. Let AP, DQ meet at X and BP, CQ meet at Y. Prove that Area of ΔADX + Area of ΔBCY = Area of quadrilateral PXQY.

26. Find the locus of the centroid of the tetrahedron of constant volume $64K^3$, formed by the three co-ordinates planes and a variable plane.

27. Lengths of two opposite edges of a tetrahedron are a and b. Shortest distance between these edges is d and the angle between them is θ. Prove that its volume is (1/6) abd sin θ.

28. If $\vec{r}.\vec{a} = 0$, $\vec{r}.\vec{b} = 1$ and $[\vec{r}\ \ \vec{a}\ \ \vec{b}] = 1$, $\vec{a}\vec{b} \neq 0$ and $|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = 1$ then find $\vec{r}$ in terms of $\vec{a}$ & $\vec{b}$

29. Let $\vec{u}$ & $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that $\left|(\vec{u}\times\vec{v}).\vec{w}\right| \leq \frac{1}{2}$ and the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

30. Let A is set of all possible planes passing through four vertices of given cube. Find number of ways of selecting four planes from set A, which have one common point.

31. Let OABC is a regular tetrahedron and P is any point in space. If edge length of tetrahedron is 1 unit, find the least value of $2(PA^2 + PB^2 + PC^2 + PO^2)$.

32. Find the minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = p$.

33. The lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$, $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar and these determine a single plane if $\alpha\gamma \neq \beta\delta$. Find the equation of the plane in which they lie.

34. Consider the plane E : $\vec{r} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

F is a plane containing the point A (–4, 2, 2) and parallel to E. Suppose the point B is on the plane E, such that B has a minimum distance from point A. If C(–3, 0, 4) lies in the plane F. Then find the area of ΔABC.

35. Through a point P (h, k, ℓ) a plane is drawn at right angles to OP to meet the co-ordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\left|\frac{p^5}{2hk\ell}\right|$, where O is the origin.

36. If $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ are three non collinear points and origin does not lie in the plane of the points A, B and C, then for any point $P(\vec{p})$ in the plane of the ΔABC, prove that ;

(i) $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{p}.(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$

(ii) A point $\vec{v}$ is on plane of ΔABC such that vector $\overrightarrow{ov}$ is $\perp$ to plane of ΔABC. Then show that $\vec{v} = \frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}](\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{4\Delta^2}$, where $\vec{\Delta}$ is the vector area of the Δ ABC.

37. Prove that the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) and having radius as small as possible is $3\sum x^2 - 2\sum x - 1 = 0$.

38. Prove that the line $\frac{x}{1} = \frac{y}{1} = \frac{z-1}{-2}$ lies in the plane x + y + z = 1. Find the lines in the plane through the point (0, 0, 1) which are inclined at an angle $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$ with the line.

39. Find the equation of the sphere which has centre at the origin and touches the line $2(x + 1) = 2 - y = z + 3$.

40. A mirror and a source of light are situated at the origin O and at a point on OX (x-axis) respectively. A ray of light from the source strikes the mirror and is reflected. If the Drs of the normal to the plane are 1, –1, 1, then find d.c's of the reflected ray.

41. A variable plane $\ell x + my + nz = p$ (where ℓ, m, n are direction cosines) intersects with co-ordinate axes at points A, B and C respectively show that the foot of normal on the plane from origin is the orthocentre of triangle ABC and hence find the coordinates of circumcentre of triangle ABC.

42. A rectangle whose vertices are (5,3,–3), (5,9,9), (0,5,11) and (0,–1,–1) is rotated about its diagonal (whose direction cosines are $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$ in such a way that new position of rectangle is perpendicular to its old position find the coordinates of new position of the vertices whose position is changed.

43 A solid sphere 'S' present in space (above X-Y plane) whose equation is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

(i) A light source lies on the line $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ above X-Y plane at infinite distance from X-Y plane. If a = c and b = 0 then find the equation of the boundary of shadow of 'S' on X-Y plane.

(ii) A light source is at (5,0,3), a = c = 2, b = 0, r = 1. What is the locus name of boundary of shadow of 'S' on X-Y plane.

Answers

4. 0 **7.** $\lambda = \frac{1}{2}$ **15.** **(ii)** (2, 8, – 3) & (0, 1, 2) ; $\sqrt{78}$; $\frac{x-2}{2} = \frac{y-8}{7} = \frac{z+3}{-5}$

16. $\frac{1}{\sqrt{2}}$ **17.** 1 **19.** (ii) 27 **20.** $\vec{R} = 10\hat{i}$

21. $\overrightarrow{AB} - \overrightarrow{AD} = 3\frac{\vec{c}\times\vec{a}}{|\vec{a}|^2}$; $\overrightarrow{AC} = \frac{1}{3}\vec{a} + \frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2} + \frac{3(\vec{b}\times\vec{a})}{|\vec{a}|^2}$; $\overrightarrow{AD} = \frac{1}{3}\vec{a} + \frac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2} - \frac{3(\vec{c}\times\vec{a})}{|\vec{a}|^2}$

23. $-|\vec{a}|^{2018}$ **26.** $xyz = 6k^3$ **28.** $\vec{r} = -(\vec{a}\cdot\vec{b})\,\vec{a} + |\vec{a}|^2\,\vec{b} + (\vec{a}\times\vec{b})$ **30.** 135

31. 3 **32.** $\frac{p^2}{\sum a^2}$ **33.** $x - 2y + z = 0$ **34.** $\frac{9}{2}$

38. $\frac{x}{1+\sqrt{15}} = \frac{y}{1-\sqrt{15}} = \frac{z-1}{-2}$ and $\frac{x}{1-\sqrt{15}} = \frac{y}{1+\sqrt{15}} = \frac{z-1}{-2}$ **39.** $9(x^2 + y^2 + z^2) = 5$

40. d.c's of the reflected ray are $\left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$ **41.** $\left(\frac{p-\ell^2 p}{2\ell}, \frac{p-m^2p}{2m}, \frac{p-n^2p}{2n}\right)$

42. {(5, –3,3), (5,11,5)} or {(–3,5,–1), (8,3,9)} **43** (i) $x^2 + 2y^2 = 2r^2$ (ii) Parabola